AN INTRODUCTION

TO

CELESTIAL MECHANICS

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BY

F R MOULTON, PH D

INSTRUCTOR IN ASTRONOMY IN THE UNIVERSITY OF CHICAGO

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PREFACE

An attempt

AN attempt has been made in this volume to give a somewhat satisfactory account of many parts of Celestial Mechanics rather than an exhaustive treatment of any special part The aim has been to present the work so as to attain logical sequence, to make it progressively more difficult, and to give the various subjects the relative prominence which their scientific and educational importance deserves In short, the aim has been to prepare such a book that one who has had the necessary mathematical training may obtain from it in a relatively short time and by the easiest steps a sufficiently broad and just view of the whole subject to enable him to stop with much of real value in his possession, or to pursue to the best advantage any particular portion he may choose

In carrying out the plan of this work it has been necessary to give an introduction to the Problem of Three Bodies This is not only one of the justly celebrated problems of Celestial Mechanics, but it has become of special interest in recent times through the researches of Hill, Poincaré, and Darwin The theory of absolute perturbations is the central subject in mathematical Astronomy, and such a work as this would be inexcusably deficient if it did not give this theory a prominent place A chapter has been devoted to geometrical considerations on perturbations Although these methods are of almost no use in computing, yet they furnish in a simple manner a clear insight into the nature of the problem, and are of the highest value to beginners The fundamental principles of the analytical methods have been given with considerable completeness, but many of the details in developing the formulas have been omitted in order that the size of the book might not defeat the object for which it has been prepared The theory of orbits has not been given the unduly prominent position which it has occupied in this country, doubtless due to the influence of Watson's excellent treatise on this subject

The method of treatment has been to state all problems in advance and, where the transformations are long, to give an outline of the steps

which are to be made The expression "order of small quantities" has not been used except when applied to power series in explicit parameters, thus giving to the work all the definiteness and simplicity which are characteristic of operations with power series This is exemplified particularly in the chapter on perturbations Care has been taken to make note at all places where assumptions have been introduced or unjustified methods employed, for it is only by seeing where the points of possible weakness are that improvements can be made The frequent references throughout the text and the bibliographies at the ends of the chapters, though by no means exhaustive, are sufficient to direct one in further reading to important sources of information

This volume is the outgrowth of a course of lectures given annually by the author at the University of Chicago during the last six years These lectures have been open to Senior College students and to graduate students who have not had the equivalent of this work They have been taken by students of Astronomy, by many making Mathematics their major work, and by some who, although specializing in quite distinct lines, have desired to get an idea of the processes by means of which astronomers interpret and predict celestial phenomena Thus they have served to give many an idea of the methods of investigation and the results attained in Celestial Mechanics, and have prepared some for a detailed study extending into the various branches of modern investigations The object of the work, the subjects covered, and the methods of treatment seem to have been amply justified by this experience

Mr A C Lunn, M A, has read the entire manuscript with great care and a thorough insight into the subjects treated His numerous corrections and suggestions have added greatly to the accuracy and the method of treatment in many places Professor Ormond Stone has read the proofs of the first four chapters and the sixth His experience as an investigator and as a teacher has made his criticisms and suggestions invaluable Mr W O Beal, M A, has read the proofs of the whole book with great attention and he is responsible for many improvements The author desires to express his sincerest thanks to all of these gentlemen for the willingness and the effectiveness with which they have devoted so much of their time to this work

F R MOULTON

CHICAGO, *July*, 1902

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CHAPTER I

FUNDAMENTAL PRINCIPLES AND DEFINITIONS

1 Elements and Laws The problems of every science are expressible in certain terms which will be designated as *elements*, and depend upon certain *principles* and *laws* for their solution The elements arise from the very nature of the subject considered, and are expressed or implied in the formulation of the problems treated The principles and laws are the relations which are known or assumed to exist among the various elements They are inductions from experiments, or deductions from previously accepted principles and laws, or simply agreements

An explicit statement on the start of the type of the problems which will be treated, and an enumeration of the elements involved, and of the principles and laws which relate to them, will lead to clearness of exposition In order to obtain a complete understanding of the character of the conclusions it would be necessary to make a philosophical discussion of the reality of the elements, and of the origin and character of the principles and laws These questions cannot be entered into here owing to the difficulty and complexity of metaphysical speculations It is not to be understood that such investigations are not of value, they forever lead back to simpler and more undeniable assumptions upon which to base all reasoning

The method of procedure in this work will necessarily be to accept as true certain fundamental elements and laws without entering in detail into the questions of their reality or validity It will be sufficient to consider whether they are definitions or have been inferred from experience, and to point out that they have been abundantly verified in their applications They will be accepted with confidence, and their consequences will be derived, in the subject treated, so far as the scope and limits of the work will allow

2 Problems Treated The motions of a material body subject to a central force of any sort whatever will be briefly considered From the theorems reached, and the observed motions of the planets and their satellites, it will be shown that Newton's law of gravitation operates in the solar system Then taking it as being universal, it will be of fundamental importance in discussing the properties of the motions of the heavenly bodies in general

In particular, the motions of two free bodies starting from arbitrary initial conditions will be investigated, and then their motions when subject to disturbing influences of various sorts The essential features of perturbations arising from the action of a third body will be developed, both from a geometrical and an analytical point of view There will be two somewhat different cases One will be that in which the motion of a satellite around a planet is perturbed by the sun, and the second will be that in which the motion of one planet around the sun is perturbed by another planet

Another class of problems which will arise is the determination of the orbits of unknown bodies from the observations of their directions at different epochs, made from a body whose motion is known That is, the theories of the orbits of comets and planetoids will be based upon observations of their apparent positions made from the earth This incomplete outline of the questions to be treated is sufficient for the enumeration of the elements employed

3 Enumeration of the Principal Elements In the discussion of these various topics it will be necessary to employ the following elements

(*a*) Real *numbers*, and complex numbers incidentally in the solution of certain problems

(*b*) *Space* of three dimensions, possessing the same properties in every direction

(*c*) *Time* of one dimension, which will be taken as the independent variable

(*d*) *Mass*, having the ordinary properties of inertia, impenetrability, etc, which are postulated in elementary Physics

(*e*) *Force*, with the content that the same term has in Physics

Positive numbers arise in Arithmetic, and positive, negative, and complex numbers, in Algebra They have the same content here Space appears first as an essential element in Geometry Time appears first as an essential element in the subject of Kinematics But Kinematics may be regarded as a branch of Mathematics introduced

to avoid the metaphysical difficulties inherent in the subject of Mechanics, and to simplify the formulation of mechanical problems It might be said with justice, then, that time first appears as an essential element in physical discussions Mass and force appear first as essential elements in physical problems No definitions of these familiar elements are necessary here

4 Enumeration of Principles and Laws In representing the various magnitudes by numbers, certain agreements must be made as to what shall be considered as being positive, and what negative These agreements will be made so that the commutative, associative, and distributive laws of Algebra shall hold whether space or time is considered The axioms of ordinary Geometry will be considered as being true

The fundamental principles upon which all work in Theoretical Mechanics may be made to depend are Newton's three *Axioms*, or *Laws of Motion* The first two laws were known by Galileo and his followers, although they were for the first time announced together in all their completeness by Newton in the *Principia* The laws are as follows*

LAW I *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it*

LAW II *The change of motion is proportional to the force impressed, and takes place in the direction of the straight line in which the force acts*

LAW III *To every action there is an equal and opposite reaction, or, the mutual actions of two bodies are always equal and oppositely directed*

5 Nature of the Laws of Motion Newton calls them Axioms, or Laws of Motion, and after giving each, makes a few remarks concerning its import Later writers, among whom are Thomson and Tait†, regard them as inferences from experiences, but accept Newton's formulation of them as practically final, and adopt them in the precise form in which they are given in the *Principia* A number of Continental writers, among whom is Dr Ernest Mach,

* Other fundamental laws may be, and indeed have been, employed, but they involve more difficult mathematical principles at the very start They are such as D'Alembert's principle, Hamilton's principle, and the systems of Kirchhoff, Mach, Hertz, Boltzmann, etc

† *Natural Philosophy*, vol I Art 243

have given profound thought to the fundamental principles of Rational Mechanics, and have concluded that they are not only *inductions*, but that Newton's statement of them is redundant, and that it lacks scientific directness and simplicity. There is nothing, however, to show that Newton's laws are not strictly in accordance with every observed phenomenon; they will be accepted here without any attempt to enter into the metaphysical aspects of the problem, although the import of each will be pointed out.

6. Nature of the First Law. In the first law the statement that a body subject to no forces moves with uniform motion, may be regarded as a definition of *time*. For, if it is not, it implies that there exists some method of measuring time in which motion is not involved. Now it is a fact that in the devices actually used for measuring time this part of the law is a fundamental assumption. For example, it is assumed that the earth rotates at a uniform rate because there is no force acting upon it which changes the rotation sensibly.*

The second part of the law, which affirms that the motion is in a straight line when the body is subject to no forces, may be taken as defining a *straight line*, if it be assumed that it is possible to determine when a body is subject to no forces; or, it may be taken as showing, together with the first part, whether forces are acting or not, if it be assumed that it is possible to give an independent definition of a straight line. Either alternative leads to troublesome difficulties when an attempt is made to employ strict and consistent definitions.

7. Nature of the Second Law. In the second law the statement that the change of motion is proportional to the force impressed, may be regarded as a definition of the relation between force and matter by means of which the magnitude of a force, or the amount of matter in a body may be measured. By change of motion is meant the change of velocity multiplied by the mass of the body moved. This is usually called the *change of momentum*, and the ideas of the second law may be expressed by saying, *the change of momentum is proportional to the force impressed and takes place in the direction of the straight line in which the force acts.* Or, *the acceleration of motion of a body is directly proportional to the force to which it is subject, and inversely proportional to its mass, and takes place in the direction in which the force acts.*

It may appear at first thought that force may be measured without reference to velocity generated, and it is true in a sense. For example,

* See memoir by R. S. Woodward, *Ast. Journ.*, No. 502.

the force with which gravity draws a body downward is frequently measured by the stretching of a coiled spring, or the intensity of magnetic action, by the torsion of a fiber But it will be noticed in all cases of this kind that the law of reaction of the machine has been determined in some other way This may not have been by velocities generated, but it ultimately leads back to it It is worthy of note in this connection that all the units of absolute force, as the *dyne*, contain explicitly in their definitions the idea of velocity generated *

In the statement of the second law it is implied that the effect of a force is exactly the same in whatever condition of rest or of motion the body may be, and to whatever other forces it may be subject The position of a body acted upon by a number of forces is the same at the end of a unit of time as it would be if each force acted separately for the same time Then the implication in the second law is, *if any number of forces act simultaneously on a body, whether it is at rest or in motion, each force produces the same change of momentum that it would if it alone acted on the body at rest* It is apparent that this principle leads to great simplifications of problems, for in accordance with it the effects of the various forces may be considered separately

Newton derived the *parallelogram of forces* from this law † He reasoned that as forces are measured by the accelerations which they produce, the resultant of, say, two forces should be measured by the resultant of their accelerations Since an acceleration has magnitude and direction it may be represented by a directed line, or *vector* The resultant will then be represented by the diagonal of a parallelogram, of which two adjacent sides represent the two forces

One of the most frequent applications of the parallelogram of forces is in the subject of Statics, which, in itself, does not involve the ideas of motion and time In it the idea of mass may also be entirely eliminated Newton's proof of the parallelogram of forces has been objected to on the ground that it requires the introduction of the fundamental conceptions of a much more complicated science than the one in which it is employed Among the demonstrations which avoid this objectionable feature is one due to Poisson ‡, which has for its fundamental assumption the axiom that the resultant of two equal forces applied at a point is in the line of the bisector of the angle which they make with each other Then the magnitude of the resultant is derived, and by simple processes the general law is obtained

* Appell's *Mécanique*, vol I p 95

† *Principia*, Cor I to the laws of motion

‡ *Traité de Mécanique*, vol I p 45 *et seq*

8 Nature of the Third Law The first two laws are sufficient for the determination of the motion of one body subject to any number of known forces, but another principle is needed when the investigation concerns the motion of a system of two or more bodies subject to their mutual interactions The third law of motion expresses precisely this principle It is that if one body presses against another the second resists the action of the first with the same force And, this which is not so easily conceived, if one body acts upon another through any distance, the second reacts upon the first with an equal and oppositely directed force

Suppose one can exert a given force at will, then, by the second law of motion, the relative masses of bodies can be measured since they are inversely proportional to the accelerations which equal forces generate in them When their relative masses have been found the third law can be tested by permitting various bodies to act upon each other and measuring their relative accelerations Newton made several experiments to verify the law, such as measuring the rebounds from the impact of elastic bodies, and observing the accelerations of magnets floating in basins of water* The chief difficulty in the experiments arises in eliminating forces external to the system under consideration, and evidently they cannot be completely removed Newton also concluded from a certain course of reasoning that to deny the third law would be to contradict the first*

In the scholium appended Newton made some remarks concerning an important feature of the third law This was first stated in a manner in which it could actually be expressed in mathematical symbols by D'Alembert in 1742, and has ever since been known by his name† It is essentially this When a body is subject to an acceleration, it may be regarded as exerting a force which is equal and opposite to the force by which the acceleration is produced This may be considered as being true whether the force arises from another body forming a system with the one under consideration, or has its source exterior to the system In general, in a system of any number of bodies, the resultants of all the applied forces are equal and opposite to the reactions of the respective bodies In other words, the *impressed* forces and the reactions, or the *expressed* forces, form systems which are in equilibrium for each body and for the whole system This makes the whole science of Dynamics, in form, one of Statics, and formulates the conditions so that they are expressible in mathematical

* *Principia* Scholium to the laws of motion

† See Appell's *Mecanique*, vol II chap XXIII

terms This phrasing of the third law of motion has been made the starting point for the elegant and very general investigations of Lagrange in the subject of Dynamics*

The primary purpose of fundamental principles in a science is to coordinate the various phenomena by stating in what respects their modes of occurring are common, the value of fundamental principles depends upon the completeness of the coordination of the phenomena, and upon the readiness with which they lead to the discovery of unknown facts, the characteristics of fundamental principles should be that they are self-consistent, that they are consistent with every observed phenomenon, and that they are simple and not redundant

Newton's laws coordinate the phenomena of the mechanical sciences in a remarkable manner, while their value in making discoveries is witnessed by the brilliant achievements in the physical sciences in the last two centuries compared to the slow and uncertain advances of all the ancients They are self-consistent, and are consistent with all the phenomena which have been so far observed, they are conspicuous for their simplicity, but it has been claimed by some that they are in some respects redundant One naturally wonders whether they are primary and fundamental laws of nature In view of the past evolution of scientific and philosophical ideas one should be slow in affirming that any statement represents ultimate and absolute truth The fact that several other sets of fundamental principles have been made the bases of systems of mechanics, points to the possibility that perhaps some time the Newtonian system, even though it may not be found to be in error, will be supplemented by a simpler one even in elementary books

Definitions and General Equations

9 **Rectilinear Motion, Speed, Velocity** A particle is in *rectilinear motion* when it always lies in the same straight line, and when its distance from a point in that line varies with the time It moves with *uniform speed* if it passes over equal distances in equal intervals of time, whatever their length The speed is represented by a positive number, and is measured by the distance passed over in a unit of a time The *velocity* of a particle is the directed speed with which it moves, and is positive or negative according to the direction of the motion Hence the velocity is given by the equation

$$v=\frac{s}{t} \qquad (1)$$

* *Collected Works*, vols XI and XII

Since s may be positive or negative v may be positive or negative, the speed being the numerical value of v The same value of v is obtained whatever interval of time is taken so long as the corresponding value of s is used

The speed and velocity are *variable* when the particle does not describe equal distances in equal times, and it is necessary to define in this case what the speed and velocity at any instant mean Suppose a particle passes over the distance Δs in the time Δt Suppose the interval of time Δt approaches the limit zero in such a manner that it always contains the instant t Suppose that for every Δt the corresponding Δs is taken Then the velocity at the instant t is defined as

$$v = \lim_{\Delta t = 0} \left(\frac{\Delta s}{\Delta t}\right) = \frac{ds}{dt}, \tag{2}$$

and the speed is the numerical value of $\frac{ds}{dt}$

Uniform and variable velocity may be defined analytically in the following manner The distance s, counted from a fixed point, is considered as a function of the time, and may be written

$$s = \phi(t)$$

Then the velocity may be defined by the equation

$$v = \frac{ds}{dt} = \phi'(t),$$

where $\phi'(t)$ is the derivative of $\phi(t)$ with respect to t The velocity is said to be constant, or uniform, if $\phi'(t)$ does not contain t explicitly, or, in other words, if $\phi(t)$ involves t linearly as, $\phi(t) = at + b$ It is said to be variable if the value of $\phi'(t)$ changes with t

Some agreement must be made to denote the direction of motion An arbitrary point on the line may be taken as the origin and the distances to the right counted as positive, and those to the left, negative With this convention, if the value of s determining the position of the body increases as the time increases the velocity will be taken positive, if the value of s decreases as the time increases the velocity will be taken negative Then, when v is given in magnitude and sign, the speed and direction of motion are determined

10 Acceleration in Rectilinear Motion Acceleration is the rate of change of velocity, and may be constant or variable Since the case when it is variable includes that when it is constant, it will be sufficient to consider the former The definition of acceleration at an

instant t is similar to that for velocity, and is, if the acceleration be denoted by a,

(3) $$a = \lim_{\Delta t = 0} \left(\frac{\Delta v}{\Delta t}\right) = \frac{dv}{dt}$$

By means of (2) and (3) it follows that

(4) $$a = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

There must be an agreement regarding the sign of the acceleration When the velocity increases as the time increases, the acceleration will be taken positive, when the velocity decreases as the time increases, the acceleration will be taken negative

11 **Speed and Velocities in Curvilinear Motion** The speed with which a particle moves is the rate at which it describes a curve If v represents the speed in this case, and s the arc of the curve, then

(5) $$v = \left|\frac{ds}{dt}\right|,$$

where $\left|\frac{ds}{dt}\right|$ represents the numerical value of $\frac{ds}{dt}$ As before, the velocity is the directed speed possessing the properties of vectors, and may be represented by a vector* The vector may be resolved uniquely into three components parallel to any three coordinate axes, and conversely, the three components may be compounded uniquely into the vector In other words, if the velocity is given, the components parallel to any coordinate axes are defined, and conversely, the components parallel to any coordinate axes define the velocity It is generally simplest to use rectangular axes and to employ the components of velocity parallel to them Let λ, μ, ν represent the angles between the line of motion and the x, y, and z-axes respectively Then

(6) $$\cos\lambda = \frac{dx}{ds}, \quad \cos\mu = \frac{dy}{ds}, \quad \cos\nu = \frac{dz}{ds}$$

Let v_x, v_y, v_z represent the components of velocity along the three axes That is,

(7) $$\begin{cases} v_x = v\cos\lambda = \dfrac{ds}{dt}\dfrac{dx}{ds} = \dfrac{dx}{dt}, \\[2ex] v_y = v\cos\mu = \dfrac{ds}{dt}\dfrac{dy}{ds} = \dfrac{dy}{dt}, \\[2ex] v_z = v\cos\nu = \dfrac{ds}{dt}\dfrac{dz}{ds} = \dfrac{dz}{dt} \end{cases}$$

* Consult Appell's *Mécanique*, vol I p 45 *et seq*

From these equations it follows that

$$(8) \qquad v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

There must be an agreement as to a positive and a negative direction along each of the three coordinate axes

12 Acceleration in Curvilinear Motion As in the case of velocities, it is simplest to resolve the acceleration into component accelerations parallel to the coordinate axes Constructing a notation corresponding to that used in Art 11, the following equations result

$$(9) \qquad a_x = \frac{d^2x}{dt^2}, \qquad a_y = \frac{d^2y}{dt^2}, \qquad a_z = \frac{d^2z}{dt^2}$$

The resultant acceleration is

$$(10) \qquad a = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$$

This is not, in general, equal to the acceleration along the curve, that is, to $\frac{d^2s}{dt^2}$ For, from (8) it follows that

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2},$$

whence, by differentiation,

$$(11) \qquad \frac{d^2s}{dt^2} = \frac{\frac{dx}{dt}\frac{d^2x}{dt^2} + \frac{dy}{dt}\frac{d^2y}{dt^2} + \frac{dz}{dt}\frac{d^2z}{dt^2}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2}$$

Thus, when the components of acceleration are known, the whole acceleration is given by (10), and the acceleration along the curve by (11)

13 The Components of Velocity Along and Perpendicular to the Radius Vector Suppose the path of the particle is in the xy-plane, and let the polar coordinates be r and θ Then

$$(12) \qquad x = r\cos\theta, \quad y = r\sin\theta$$

The components of velocity are therefore

$$(13) \qquad \begin{cases} \dfrac{dx}{dt} = v_x = -r\sin\theta\dfrac{d\theta}{dt} + \cos\theta\dfrac{dr}{dt}, \\[2ex] \dfrac{dy}{dt} = v_y = r\cos\theta\dfrac{d\theta}{dt} + \sin\theta\dfrac{dr}{dt} \end{cases}$$

Let QP be an arc of the curve described by the moving particle When the particle is at P, it is moving in the direction PV, and the velocity may be represented by the vector PV Let v_r and v_θ represent the components of velocity along and perpendicular to the radius vector But the resultant of the vectors v_r and v_θ is equal to the resultant of the vectors $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and therefore the sums of their projections upon any line are equal* Therefore, projecting v_r and v_θ upon the x and y-axes, it follows that

$$(14)\qquad \begin{cases} \dfrac{dx}{dt} = v_r \cos\theta - v_\theta \sin\theta, \\ \dfrac{dy}{dt} = v_r \sin\theta + v_\theta \cos\theta \end{cases}$$

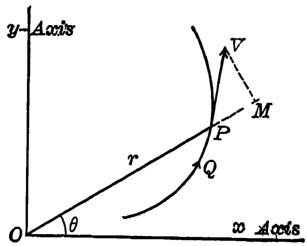

Fig 1

Comparing (13) and (14) the required components of velocity are found to be

$$(15)\qquad \begin{cases} v_r = \dfrac{dr}{dt}, \\ v_\theta = r\,\dfrac{d\theta}{dt} \end{cases}$$

The square of the speed is

$$v_r^2 + v_\theta^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

The components of velocity, v_r and v_θ, may be found in terms of the components parallel to the x and y-axes by multiplying equations (14) by $\cos\theta$ and $\sin\theta$ respectively and adding, and then by $-\sin\theta$ and $\cos\theta$ and adding The results are

$$(16)\qquad \begin{cases} v_r = \cos\theta\,\dfrac{dx}{dt} + \sin\theta\,\dfrac{dy}{dt}, \\ v_\theta = -\sin\theta\,\dfrac{dx}{dt} + \cos\theta\,\dfrac{dy}{dt} \end{cases}$$

* See Appell's *Mécanique*, vol I chap I

14 The Components of Acceleration The derivatives of (13) are

$$
(17) \qquad \begin{cases} a_x = \dfrac{d^2x}{dt^2} = \left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right]\cos\theta - \left[r\,\dfrac{d\,\theta}{dt^2} + 2\,\dfrac{dr}{dt}\dfrac{d\theta}{dt}\right]\sin\theta, \\ a_y = \dfrac{d^2y}{dt^2} = \left[r\,\dfrac{d\,\theta}{dt^2} + 2\,\dfrac{dr}{dt}\dfrac{d\theta}{dt}\right]\cos\theta + \left[\dfrac{d^2r}{dt} - r\left(\dfrac{d\theta}{dt}\right)^2\right]\sin\theta \end{cases}
$$

Let a_r and a_θ represent the components of acceleration along and perpendicular to the radius vector As in Art 13, it follows from the composition and resolution of vectors that

$$
(18) \qquad \begin{cases} a_x = a_r\cos\theta - a_\theta\sin\theta, \\ a_y = a_r\sin\theta + a_\theta\cos\theta \end{cases}
$$

Comparing (17) and (18) it follows that

$$
(19) \qquad \begin{cases} a_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right), \\ a_\theta = r\,\dfrac{d^2\theta}{dt^2} + 2\,\dfrac{dr}{dt}\dfrac{d\theta}{dt} = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\,\dfrac{d\theta}{dt}\right) \end{cases}
$$

The components of acceleration along and perpendicular to the radius vector in terms of the components parallel to the x and y-axes are found from (17) to be

$$
(20) \qquad \begin{cases} a_r = \cos\theta\,\dfrac{d\,x}{dt^2} + \sin\theta\,\dfrac{d\,y}{dt}, \\ a_\theta = -\sin\theta\,\dfrac{d^2x}{dt} + \cos\theta\,\dfrac{d^2y}{dt^2} \end{cases}
$$

By similar processes the components of velocity and acceleration parallel to any lines may be found

15 Application to a Particle Moving in a Circle with Uniform Speed Suppose the particle moves with uniform speed

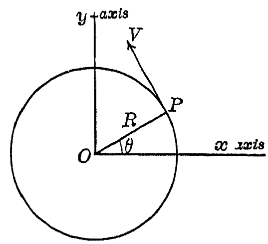

Fig 2

in a circle around the origin as center, it is required to determine the components of velocity and acceleration parallel to the x and y-axes, and parallel and perpendicular to the radius Let R represent the radius of the circle, then

$$
x = R\cos\theta, \qquad y = R\sin\theta
$$

Since the speed is uniform the angle θ is proportional to the time, or $\theta = ct$ The coordinates become

$$(21) \qquad x = R\cos(ct), \qquad y = R\sin(ct)$$

Since $\frac{d\theta}{dt} = c$ and $\frac{dR}{dt} = 0$, the components of velocity parallel to the x and y-axes are found from (13) to be

$$(22) \qquad v_x = -Rc\sin(ct), \qquad v_y = Rc\cos(ct)$$

From (15) it is found that

$$(23) \qquad v_r = 0, \qquad v_\theta = Rc$$

The components of acceleration parallel to the x and y-axes are given by (17), and are

$$(24) \qquad \begin{cases} a_x = -Rc^2\cos(ct), \\ a_y = -Rc^2\sin(ct) \end{cases}$$

From (19) it is found that

$$(25) \qquad a_r = -Rc^2, \qquad a_\theta = 0$$

It will be observed that, although the speed is uniform in this case, the velocity with respect to fixed axes is not constant, and the acceleration is not zero If it be assumed that an exterior force is the only cause of the change of motion, or of acceleration of a particle, then it follows that a particle cannot move in a circle with uniform speed unless it is subject to some force It follows also from (25) and the second law of motion that the force continually acts in a line passing through the center of the circle

16 The Areal Velocity The rate at which the radius vector from a fixed point to the moving particle describes a surface is called

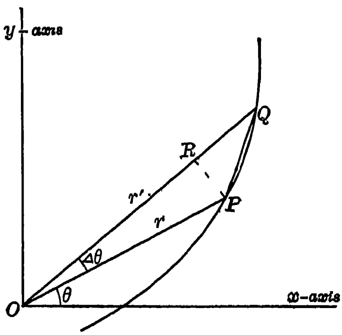

Fig 3

the *areal velocity* Suppose the particle moves in the xy-plane Let ΔA represent the area of the triangle OPQ swept over by the radius vector in the interval of time Δt

Then
$$\Delta A = \frac{r' \; r}{2} \sin(\Delta\theta),$$
whence

(26)
$$\frac{\Delta A}{\Delta t} = \frac{r' \; r}{2} \; \frac{\sin(\Delta\theta)}{\Delta\theta} \; \frac{\Delta\theta}{\Delta t}$$

As the angle $\Delta\theta$ diminishes the ratio of the area of the triangle to that of the sector approaches unity as a limit The limit of r' is r, and of $\frac{\sin(\Delta\theta)}{\Delta\theta}$, unity Equation (26) gives, passing to the limit in both members,

(27)
$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

as the expression for the areal velocity Changing to rectangular coordinates by the substitution
$$r = \sqrt{x^2 + y}, \qquad \tan\theta = \frac{y}{x},$$
equation (27) becomes

(28)
$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

If the motion is not in the xy-plane the projections of the areal velocity upon the three fundamental planes are used They are respectively

(29)
$$\begin{cases} \dfrac{dA_{xy}}{dt} = \dfrac{1}{2}\left(x\dfrac{dy}{dt} - y\dfrac{dx}{dt}\right), \\ \dfrac{dA_{yz}}{dt} = \dfrac{1}{2}\left(y\dfrac{dz}{dt} - z\dfrac{dy}{dt}\right), \\ \dfrac{dA_{zx}}{dt} = \dfrac{1}{2}\left(z\dfrac{dx}{dt} - x\dfrac{dz}{dt}\right) \end{cases}$$

In certain motions of bodies in mechanical problems the areal velocity is constant if the origin is properly chosen In this case it is said that the body obeys the law of areas with respect to the origin That is,
$$r \frac{d\theta}{dt} = \text{constant}$$

It follows from this equation and (19) that in this case
$$a_\theta = 0,$$
that is, the acceleration perpendicular to the radius vector is zero

17 Application to Motion in an Ellipse Suppose the particle moves in an ellipse whose semi-axes are a and b in such a manner that it obeys the law of areas with respect to the center of the ellipse as origin, it is required to find the components of acceleration along and perpendicular to the radius vector The equation of the ellipse may be written

$$(30) \qquad x = a\cos\phi, \qquad y = b\sin\phi,$$

for, if ϕ is eliminated the ordinary equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is found It follows from (30) that

$$(31) \qquad \frac{dx}{dt} = -a\sin\phi\,\frac{d\phi}{dt}, \qquad \frac{dy}{dt} = b\cos\phi\,\frac{d\phi}{dt}$$

Substituting (30) and (31) in the expression for the law of areas,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c,$$

it becomes

$$\frac{d\phi}{dt} = \frac{c}{ab} = c_1$$

The integral of this equation is

$$\phi = c_1 t + c_2,$$

and if $\phi = 0$ when $t = 0$, then $c_2 = 0$ and $\phi = c_1 t$

Substituting this value of ϕ in (30), it is found that

$$\begin{cases} \dfrac{d^2x}{dt^2} = -c_1^2 a\cos\phi = -c_1^2 x, \\[2ex] \dfrac{d^2y}{dt^2} = -c_1^2 b\sin\phi = -c_1^2 y \end{cases}$$

Substituting these values of the derivatives in (20), and writing in place of $\cos\theta$ and $\sin\theta$, $\frac{x}{r}$ and $\frac{y}{r}$ respectively, the components of acceleration are found to be

$$\begin{cases} a_r = -c_1^2 r, \\ a_\theta = 0 \end{cases}$$

I PROBLEMS

1 A particle moves with uniform speed along a helix traced on a cylinder whose radius is R, find the components of velocity and acceleration parallel to the x, y, and z axes The equations of the helix are

$$x = R\cos\omega, \quad y = R\sin\omega, \quad z = h\omega$$

Ans $$\begin{cases} v_x = -Rc\sin(ct), & v_y = Rc\cos(ct), & v_z = hc, \\ a_x = -Rc^2\cos(ct), & a_y = -Rc^2\sin(ct), & a_z = 0 \end{cases}$$

2 A particle moves in the ellipse whose parameter and eccentricity are p and e with uniform angular speed with respect to one of the foci as origin, it is required to find the components of velocity and acceleration along and perpendicular to the radius vector and parallel to the x and y axes in terms of the radius vector and the time

Ans $$\begin{cases} v_r = \dfrac{ec}{p}\, r^2\sin(ct), \qquad v_\theta = rc, \\ v_x = -cr\sin(ct) + \dfrac{ec}{2p}\, r^2\sin(2ct), \qquad v_y = cr\cos(ct) + \dfrac{ec}{p}\, r^2\sin^2(ct), \\ a_r = \dfrac{ec^2}{p}\, r^2\cos ct + \dfrac{2e^2c^2}{p^2}\, r^3\sin^2(ct) - c^2 r, \\ a_\theta = \dfrac{2ec^2}{p}\, r^2\sin(ct), \\ a_x = -c^2 r\cos(ct) + \dfrac{ec^2}{p}\, r^2 - \dfrac{3ec^2}{p}\, r^2\sin^2(ct) + \dfrac{2e^2c^2}{p}\, r^3\sin^2(ct)\cos(ct), \\ a_y = -c^2 r\sin ct + \dfrac{3ec^2}{2p}\, r^2\sin(2ct) + \dfrac{2e^2c^2}{p^2}\, r^3\sin^3(ct) \end{cases}$$

3 A particle moves in an ellipse in such a manner that it obeys the law of areas with respect to one of the foci as origin, it is required to find the components of velocity and acceleration along and perpendicular to the radius vector and parallel to the axes in terms of the coördinates

Ans $$\begin{cases} v_r = \dfrac{eA}{p}\sin\theta, \qquad v_\theta = \dfrac{A}{r}, \\ v_x = \dfrac{eA}{2p}\sin 2\theta - \dfrac{A\sin\theta}{r}, \qquad v_y = \dfrac{eA}{p}\sin^2\theta + \dfrac{A\cos\theta}{r}, \\ a_r = -\dfrac{A}{p}\,\dfrac{1}{r^2}, \qquad a_\theta = 0, \\ a_x = -\dfrac{A^2}{p}\,\dfrac{\cos\theta}{r}, \qquad a_y = -\dfrac{A^2}{p}\,\dfrac{\sin\theta}{r^2} \end{cases}$$

4 The accelerations along the x and y axes are the derivatives of the velocities along these axes, why are not the accelerations along and perpendicular to the radius vector given by the derivatives of the velocities in these respective directions?

18 Center of Mass of n Equal Particles The center of mass of a system of equal particles is defined as that point whose distance from any plane is equal to the average distance of all of the particles from that plane This must be true then for the three reference planes Let x_1, y_1, z_1, x_2, y_2, z_2, etc represent the coordinates of the various particles, and $\bar{x}$, $\bar{y}$, $\bar{z}$ the coordinates of their center of mass, then by the definition

$$(32)\qquad \begin{cases} \bar{x} = \dfrac{x_1 + x_2 + \quad + x_n}{n} = \dfrac{\sum\limits_{i=1}^{n} x_i}{n}, \\[2ex] \bar{y} = \dfrac{y_1 + y_2 + \quad + y_n}{n} = \dfrac{\sum\limits_{i=1}^{n} y_i}{n}, \\[2ex] \bar{z} = \dfrac{z_1 + z_2 + \quad + z_n}{n} = \dfrac{\sum\limits_{i=1}^{n} z_i}{n} \end{cases}$$

Suppose the mass of each particle is m, and let M represent the mass of the whole system, or $M = nm$ Multiplying the numerators and denominators by m, equations (32) become

$$(33)\qquad \begin{cases} \bar{x} = \dfrac{\sum\limits_{i=1}^{n} mx_i}{M}, \\[2ex] \bar{y} = \dfrac{\sum\limits_{i=1}^{n} my_i}{M}, \\[2ex] \bar{z} = \dfrac{\sum\limits_{i=1}^{n} mz_i}{M} \end{cases}$$

19 Center of Mass of Unequal Particles There are two cases, (*a*) when the masses are not incommensurable, (*b*) when the masses are incommensurable

(*a*) Select a unit m in terms of which all the n masses can be expressed integrally Suppose the first mass is n_1m, the second n_2m, etc and let $n_1m = m_1$, $n_2m = m_2$, etc The system may be thought of as made up of $n_1 + n_2$ particles each of mass m, and consequently by Art 18

$$(34)\quad \begin{cases} \bar{x} = \dfrac{\sum\limits_{i=1}^{n} mn_i x_i}{\sum\limits_{i=1}^{n} mn_i} = \dfrac{\sum\limits_{i=1}^{n} m_i x_i}{M}, \\ \bar{y} = \dfrac{\sum\limits_{i=1}^{n} mn_i y_i}{\sum\limits_{i=1}^{n} mn_i} = \dfrac{\sum\limits_{i=1}^{n} m_i y_i}{M}, \\ \bar{z} = \dfrac{\sum\limits_{i=1}^{n} mn_i z_i}{\sum\limits_{i=1}^{n} mn_i} = \dfrac{\sum\limits_{i=1}^{n} m_i z_i}{M} \end{cases}$$

(*b*) Select an arbitrary unit m smaller than any one of the n masses They will be expressible in terms of it plus certain remainders Neglecting the remainders equations (34) are true Take as a new unit any submultiple of m and the remainders will remain the same or be diminished, depending on their magnitudes The submultiple of m may be taken so small that every remainder is smaller than any assigned quantity Equations (34) continually hold where the m_i are the masses of the bodies minus the remainders As the submultiples of m approach zero as a limit, the remainders approach zero as a limit, and the expressions (34) approach as limits the expressions in which the m_i are the actual masses of the particles Equations (34) are therefore true in general

In order to complete the work it is necessary to show that, if the definition is fulfilled for the three reference planes, it is also fulfilled for every other plane It is to be observed that the yz-plane, for example, may be brought into any position whatever by a change of origin and a succession of rotations of the coordinate system around the various axes It will be necessary to show then that equations (34) hold true, (1) after a change of origin, and (2) after a rotation around one of the axes

(1) Transfer the origin along the x-axis through the distance a The substitution is $x = x' + a$, and the first equation of (34) becomes

$$\bar{x}' + a = \frac{\sum\limits_{i=1}^{n} m_i (x_i' + a)}{M} = \frac{\sum\limits_{i=1}^{n} m_i x_i'}{M} + \frac{a \sum\limits_{i=1}^{n} m_i}{M},$$

whence $$\bar{x}' = \frac{\sum\limits_{i=1}^{n} m_i x_i'}{M},$$

which has the same form as before

(2) Rotate the x and y-axes around the z-axis through the angle θ The substitution is

$$\begin{cases} x = x' \cos\theta - y' \sin\theta, \\ y = x' \sin\theta + y' \cos\theta \end{cases}$$

The first two equations of (34) become

$$\begin{cases} \bar{x}' \cos\theta - \bar{y}' \sin\theta = \cos\theta \dfrac{\sum\limits_{i=1}^{n} m_i x_i'}{M} - \sin\theta \dfrac{\sum\limits_{i=1}^{n} m_i y_i'}{M}, \\ \bar{x}' \sin\theta + \bar{y}' \cos\theta = \sin\theta \dfrac{\sum\limits_{i=1}^{n} m_i x_i'}{M} + \cos\theta \dfrac{\sum\limits_{i=1}^{n} m_i y_i'}{M} \end{cases}$$

Solving these equations it is found that

$$\bar{x}' = \frac{\sum\limits_{i=1}^{n} m_i x_i'}{M}, \qquad \bar{y}' = \frac{\sum\limits_{i=1}^{n} m_i y_i'}{M}$$

Therefore equations (34) hold true with the planes in the new positions, and consequently the condition is fulfilled for every plane

20 The Center of Gravity The members of a system of particles which are near together at the surface of the earth are subject to forces downward which are sensibly parallel and proportional

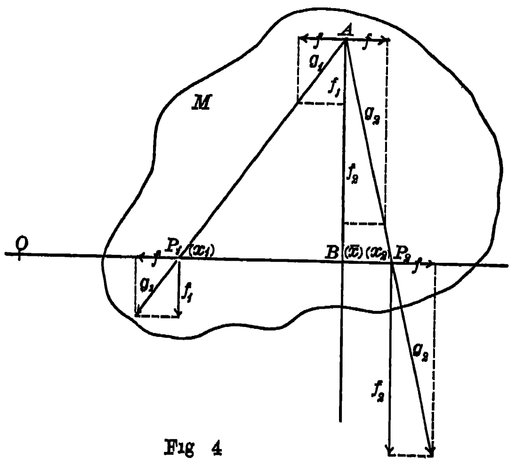

Fig 4

to their respective masses The *weight*, or *gravity*, of a particle will be defined as the intensity of the vertical force f, which is the product of the mass m of the particle and its acceleration g The *center of gravity* of the system is such a point that, if the members of the system were rigidly connected and the sum of all the forces applied at this point, then the effect on the motion of the system would be the same as that of the original forces for all orientations of the system

It will now be shown that the center of gravity coincides with the center of mass. Consider two parallel forces f_1 and f_2 acting upon the rigid system M at the points P_1 and P_2. Resolve these two forces into the components f and g_1, and f and g_2 respectively. The components f, being equal and opposite, destroy each other. Then the components g_1 and g_2 may be regarded as acting at A. Resolve them again so that the oppositely directed components shall be equal and lie in a line parallel to P_1P_2, then the other components will lie in the same line AB, which is parallel to the direction of the original forces f_1 and f_2, and will be equal respectively to f_1 and f_2. Therefore the resultant of f_1 and f_2 is equal to $f_1 + f_2$ in magnitude. From similar triangles

$$\frac{f_1}{f} = \frac{AB}{P_1B}, \qquad \frac{f_2}{f} = \frac{AB}{P_2B},$$

whence, by division,

$$\frac{f_1}{f_2} = \frac{P_2B}{P_1B} = \frac{x_2 - x}{x - x_1}.$$

The solution for x gives

$$x = \frac{f_1x_1 + f_2x_2}{f_1 + f_2}.$$

If the resultant of these two forces be united with a third force f_3, the point where their sum may be applied with the same effects is found in a similar manner to be given by

$$x = \frac{f_1x_1 + f_2x_2 + f_3x_3}{f_1 + f_2 + f_3},$$

and so on for any number of forces. Similar equations are true for parallel forces acting in any other direction.

Suppose there are n particles m_i subject to n parallel forces f_i due to the attraction of the earth. The coordinates of their center of gravity with respect to the origin are given by

$$(35)\qquad \left\{\begin{aligned} \bar{x} &= \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} g m_i x_i}{\sum_{i=1}^{n} g m_i} = \frac{\sum_{i=1}^{n} m_i x_i}{M}, \\ \bar{y} &= \frac{\sum_{i=1}^{n} m_i y_i}{M}, \\ \bar{z} &= \frac{\sum_{i=1}^{n} m_i z_i}{M}. \end{aligned}\right.$$

The center of gravity is thus seen to be coincident with the center of mass; nevertheless it is much less general than the latter, since the system must be in such a position that the accelerations to which the various members are subject shall be equal and parallel in order that it may be defined. Euler (1707—1783) proposed the designation of *center of inertia* for this point.

21. Center of Mass of a Continuous Body. If the particles become more and more numerous and nearer together the system approaches a continuous body as a limit. In the case of the ordinary bodies of mechanics the particles are innumerable and indistinguishably close together; on this account such bodies are treated as continuous masses. For continuous masses, therefore, the limits of expressions (34), as m_i approaches zero, must be taken. At the limit m becomes dm and the summation becomes the definite integral. The equations are therefore

$$(36)\qquad \begin{cases} \bar{x} = \dfrac{\int x\,dm}{\int dm}, \\[2ex] \bar{y} = \dfrac{\int y\,dm}{\int dm}, \\[2ex] \bar{z} = \dfrac{\int z\,dm}{\int dm}, \end{cases}$$

where the integrals are to be extended throughout the whole body.

When the body is homogeneous the *density* is the quotient of any portion of the mass divided by its volume. When the body is not homogeneous the *mean density* is the quotient of the whole mass divided by the whole volume. The density at any point is the limit of the mean density of a volume including the point in question when this volume approaches zero as a limit. If the density be represented by σ, the element of mass is in rectangular coordinates

$$dm = \sigma\,dx\,dy\,dz.$$

Then equations (36) become

$$(37)\qquad \begin{cases} \bar{x} = \dfrac{\iiint \sigma x\,dx\,dy\,dz}{\iiint \sigma\,dx\,dy\,dz}, \\[2ex] \bar{y} = \dfrac{\iiint \sigma y\,dx\,dy\,dz}{\iiint \sigma\,dx\,dy\,dz}, \\[2ex] \bar{z} = \dfrac{\iiint \sigma z\,dx\,dy\,dz}{\iiint \sigma\,dx\,dy\,dz}. \end{cases}$$

The limits of the integrals depend upon the shape of the body, and σ must be expressed as a function of the coordinates.

In certain problems the integrations are performed more simply if polar coordinates are employed The element of mass is

$$dm = \sigma \ \overline{ab} \ \overline{bc} \ \overline{cd}$$

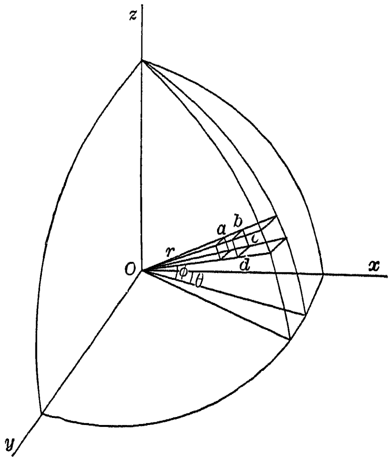

Fig 5

It is seen from the figure that

$$\begin{cases} \overline{ab} = dr, \\ \overline{bc} = r\,d\phi, \\ \overline{cd} = r \cos\phi\, d\theta \end{cases}$$

Therefore

$$(38) \qquad dm = \sigma r^2 \cos\phi\, d\phi\, d\theta\, dr,$$

and

$$(39) \qquad \begin{cases} x = r \cos\phi \cos\theta, \\ y = r \cos\phi \sin\theta, \\ z = r \sin\phi \end{cases}$$

Therefore equations (36) become

$$(40) \qquad \begin{cases} \bar{x} = \dfrac{\iiint \sigma r^3 \cos^2\phi \cos\theta\, d\phi\, d\theta\, dr}{\iiint \sigma r^2 \cos\phi\, d\phi\, d\theta\, dr}, \\ \bar{y} = \dfrac{\iiint \sigma r^3 \cos^2\phi \sin\theta\, d\phi\, d\theta\, dr}{\iiint \sigma r^2 \cos\phi\, d\phi\, d\theta\, dr}, \\ \bar{z} = \dfrac{\iiint \sigma r^3 \sin\phi \cos\phi\, d\phi\, d\theta\, dr}{\iiint \sigma r^2 \cos\phi\, d\phi\, d\theta\, dr} \end{cases}$$

The density σ must be expressed as a function of the coordinates, and the limits must be taken so that the whole body is included If the body is a line or a surface the equations admit of important simplifications

22 Planes and Axes of Symmetry If a homogeneous body is symmetrical with respect to any plane, the center of mass is in that plane This plane is called a *plane of symmetry* If a homogeneous body is symmetrical with respect to two planes, the center of mass is in the line of their intersection This line is called an *axis of symmetry* If a homogeneous body is symmetrical with respect to three planes intersecting in a point, the center of mass is at that point From the consideration of the planes and axes of symmetry the centers of mass of many of the simple figures can be inferred without employing the methods of integration

23 Application to a Heterogeneous Cube Suppose the density varies directly as the square of the distance from one of the faces of the cube Take the origin at one of the corners and the axes so that the yz-plane is the face of zero density Then $\sigma = kx^2$, where k is the density at unit distance Suppose the edge of the cube equals a, then equations (37) become

$$\left\{\begin{aligned} \bar{x} &= \frac{k\int_0^a\int_0^a\int_0^a x^3\,dx\,dy\,dz}{k\int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz}, \\ \bar{y} &= \frac{k\int_0^a\int_0^a\int_0^a x^2y\,dx\,dy\,dz}{k\int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz}, \\ \bar{z} &= \frac{k\int_0^a\int_0^a\int_0^a x^2z\,dx\,dy\,dz}{k\int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz} \end{aligned}\right.$$

These equations become, after integrating and substituting the limits,

$$\bar{x} = \frac{3a}{4}, \qquad \bar{y} = \frac{a}{2}, \qquad \bar{z} = \frac{a}{2}$$

24 Application to the Octant of a Sphere Suppose the sphere is homogeneous and that the density equals unity It is

preferable to use polar coordinates in this example, although it is by no means necessary Either (37) or (40) may be used in any problem, and the choice should be regulated by the form that the limits take in the two cases It is desirable to have them all constants so far as they may be made so Equations (40) become, if the origin is taken at the center of the sphere, and if the radius is a,

$$\left\{\begin{aligned} \bar{x} &= \frac{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\cos^2\phi\cos\theta\, d\phi\, d\theta\, dr}{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^2\cos\phi\, d\phi\, d\theta\, dr}, \\ \bar{y} &= \frac{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\cos\phi\sin\theta\, d\phi\, d\theta\, dr}{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^2\cos\phi\, d\phi\, d\theta\, dr}, \\ \bar{z} &= \frac{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\sin\phi\cos\phi\, d\phi\, d\theta\, dr}{\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{a} r^2\cos\phi\, d\phi\, d\theta\, dr} \end{aligned}\right.$$

Since the mass of a homogeneous sphere with radius a and density unity is $\frac{4}{3}\pi a^3$, each of the denominators of these expressions equals $\frac{1}{6}\pi a^3$ This may at once be verified by integration Integrating the numerators with respect to ϕ and substituting the limits, the equations become

$$\left\{\begin{aligned} \bar{x} &= \frac{\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\cos\theta\, d\theta\, dr}{\frac{\pi}{6}a^3}, \\ \bar{y} &= \frac{\frac{\pi}{4}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\sin\theta\, d\theta\, dr}{\frac{\pi}{6}a^3}, \\ \bar{z} &= \frac{\frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{a} r^3\, d\theta\, dr}{\frac{\pi}{6}a^3} \end{aligned}\right.$$

Integrating with respect to θ, these equations give

$$\left\{\begin{aligned} \bar{x} &= \frac{\frac{\pi}{4}\int_0^a r^3\,dr}{\frac{\pi}{6}a^3}, \\ \bar{y} &= \frac{\frac{\pi}{4}\int_0^a r^3\,dr}{\frac{\pi}{6}a^3}, \\ \bar{z} &= \frac{\frac{\pi}{4}\int_0^a r^3\,dr}{\frac{\pi}{6}a^3}, \end{aligned}\right.$$

and the integration with respect to r gives finally

$$\bar{x} = \bar{y} = \bar{z} = \tfrac{3}{8}a$$

As nearly all the masses occurring in astronomical problems are spheres or oblate spheroids with three planes of symmetry, the applications of the formulas just given are extremely simple, and no more examples need be solved

II PROBLEMS

1 Find the center of mass of a fine straight wire of length R whose density varies directly as the nth power of the distance from one end

Ans $\dfrac{n+1}{n+2}R$ from the end of zero density

2 Find the coördinates of the center of mass of a fine wire of uniform density bent into a quadrant of a circle of radius R

Ans $\bar{x} = \bar{y} = \dfrac{2R}{\pi}$,

where the origin is at the center of the circle

3 Find the coordinates of the center of mass of a thin plate of uniform density, having the form of a quadrant of an ellipse whose semi axes are a and b

$$Ans \quad \begin{cases} \bar{x}=\dfrac{4a}{3\pi}, \\ \bar{y}=\dfrac{4b}{3\pi} \end{cases}$$

4 Find the coordinates of the center of mass of a thin plate of uniform density, having the form of a complete loop of the lemniscate whose equation is $r^2=a \cos 2\theta$

$$Ans \quad \begin{cases} \bar{x}=\dfrac{\pi a}{2^{\frac{5}{2}}}, \\ \bar{y}=0 \end{cases}$$

5 Find the coordinates of the center of mass of an octant of an ellipsoid of uniform density whose semi axes are a, b, c

$$Ans \quad \begin{cases} \bar{x}=\dfrac{3a}{8}, \\ \bar{y}=\dfrac{3b}{8}, \\ \bar{z}=\dfrac{3c}{8} \end{cases}$$

6 Find the coordinates of the center of mass of an octant of a sphere of radius R whose density varies directly as the nth power of the distance from the center

$$Ans \quad \bar{x}=\bar{y}=\bar{z}=\frac{n+3}{n+4}\,\frac{R}{2}$$

7 Find the coordinates of the center of mass of a paraboloid of revolution cut off by a plane perpendicular to its axis

$Ans \quad \begin{cases} \bar{x}=\frac{2}{3}h, \\ \bar{y}=\bar{z}=0, \end{cases}$ where h is the distance from the vertex of the paraboloid to the plane

8 Find the coordinates of the center of mass of a right circular cone whose height is h and whose radius is R

Historical Sketch from Ancient Times to Newton

25 The Two Divisions of the History The history of the development of Celestial Mechanics is naturally divided into two distinct parts The one is concerned with the progress of knowledge about the purely formal aspect of the universe, the natural divisions of time, the configurations of the constellations, and the determination of the paths and periods of the planets in their motions, the other treats of the efforts at, and the success in, attaining correct ideas regarding the physical aspects of natural phenomena, the fundamental properties of force, matter, space, and time, and in particular, the way in which they are related It is true that these two lines in the development of astronomical science have not always been kept distinct by those who have cultivated them, indeed, on the contrary, they have often been so intimately associated that the speculations in the latter have influenced unduly the conclusions in the former While it is clear that the two kinds of investigation should be kept distinct in the mind of the investigator, it is equally clear that they should be constantly employed as checks upon each other The object of the next two articles will be to trace, in as few words as possible, the development of these two lines of progress of the science of Celestial Mechanics from the times of the early Greek Philosophers to the time when Newton applied his transcendent genius to the analysis of the elements involved, and again to their synthesis into one of the sublimest products of the human mind

26 Formal Astronomy The first division, which is concerned with phenomena apart from their causes, will be termed Formal Astronomy The day, the month, and the year are such obvious natural divisions of time that they must have been noticed by the most primitive peoples The determination of the relations among these periods required something of the scientific spirit necessary for careful observations, yet, in the very dawn of Chaldean and Egyptian history they appear to have been known with a considerable degree of accuracy The records left by these peoples of their earlier civilizations are so meager that little is known with certainty regarding their achievements The authentic history of Astronomy actually begins with the Greeks, who, deriving their first knowledge and inspiration from the Egyptians, pursued the subject with the enthusiasm and acuteness which was characteristic of the Greek race

Thales (640—546 B C), of Miletus, went to Egypt for instruction, and on his return founded the Ionian School of Astronomy and Philosophy Some idea of the advancement made by the Egyptians may be gathered from the fact that he taught the sphericity of the earth, the obliquity of the ecliptic, the causes of eclipses, and, according to Herodotus, predicted the eclipse of the sun of 585 B C According to Laertius he was the first to determine the length of the year It is fair to assume that he borrowed much of his information from Egypt

Anaximander (611—545 B C), a friend, and probably a pupil of Thales, constructed geographical maps, and is credited with having invented the gnomon

Pythagoras (569—470 B C) travelled widely in Egypt and Chaldea, penetrating Asia even to the banks of the Ganges On his return he went to Sicily and founded a School of Astronomy and Philosophy He taught that the earth both rotates and revolves, and that the comets as well as the planets move in orbits around the sun He is credited with being the first to maintain that the same planet, Venus, was both evening and morning star at different times

Meton (about 465—385 B C) brought to the notice of the learned men of Hellas the cycle of 19 years, nearly equalling 235 lunations, which has since been known as the Metonic cycle The still more accurate Callipic cycle consists of four Metonic cycles, less one day

Aristotle (384—322 B C) defended the idea of the globular form of the earth with many of the arguments which are used at the present time

The earliest writings which have been handed down to modern times, as those of Hesiod and Homer, contain frequent references to the constellations It is remarkable that, although the outlines of most of the constellations are arbitrary, the divisions used by the Greeks, the Chaldeans, and to a considerable extent by the East Indians are practically identical Eudoxus (about 409—356 B C) wrote a description of the constellations

The next important event in the development of the science of Astronomy was the founding of the Alexandrian School by Ptolemy Soter (?—283 B C), one of Alexander's Generals, who became ruler of Egypt upon the death of his master Here the earliest systematic observations were made, and the first observers were Arystillus and Timocares (about 300 B C) Their observations enabled Hipparchus and Ptolemy to make some of their greatest discoveries

Aristarchus (310—250 B C) wrote an important work entitled *Magnitudes and Distances* In it he calculated from the time which the earth is in quadrature as seen from the moon that the latter is about one-nineteenth as distant from the earth as the sun

Eratosthenes (275—194 B C) made a catalogue of 475 of the brightest stars, and is famous for having determined the size of the earth from the measurement of the difference in latitude and the distance apart of Syene, in Upper Egypt, and Alexandria

Hipparchus (190—120 B C), a native of Bithynia, who observed at Rhodes and possibly at Alexandria, was the greatest astronomer of antiquity He added to zeal and skill as an observer, the accomplishments of a mathematician Following Euclid (about 330—275 B C) at Alexandria, he developed the important science of Spherical Trigonometry He located places on the earth by their Longitudes and Latitudes and the stars by their Right Ascensions and Declinations He was led by the appearance of a temporary star to make a catalogue of 1080 fixed stars He measured the length of the tropical year, the length of the month from eclipses, the motion of the moon's nodes and that of her apogee, he was the author of the first solar tables, he discovered the precession of the equinoxes, and made extensive observations of the planets The works of Hipparchus are known only indirectly, his own writings having been lost at the time of the destruction of the great Alexandrian library by the Saracens under Omar, in 640 A D

With the assistance of the philosopher Sosigenes, Julius Caesar reformed the Roman calendar in the year 46 B C by the addition of eighty days to that year, and the decree that every fourth year thereafter should consist of 366 days The outstanding errors in this system were corrected still further by an edict of Pope Gregory XIII, in 1582, when the present calendar was inaugurated

Ptolemy (100—170 A D) carried forward the work of Hipparchus faithfully and left the *Almagest* as the monument of his labors It has fortunately come down to modern times intact and contains much information of great value Ptolemy's greatest discovery is the evection of the moon's motion, which he detected by following the moon during the whole month, instead of confining his attention to certain phases as previous observers had done He discovered refraction, but is particularly famous for the system of eccentrics and epicycles which he developed to explain the apparent motions of the planets

The stationary period followed Ptolemy during which science was not cultivated by any people except the Arabs, who were imitators and commentators rather more than original investigators. In the Ninth Century the greatest Arabian astronomer, Albategnius (850—929), flourished, and a more accurate measurement of the arc of a meridian than had before that time been executed was carried out by him in the plain of Singiar, in Mesopotamia. In the Tenth Century Al-Sufi made a catalogue of stars based on his own observations. Another catalogue was made by the direction of Ulugh Beigh (1393—1449), at Samarkand, in 1433. At this time Arabian astronomy practically ceased.

Astronomy began to revive in Europe toward the end of the Fifteenth Century in the labors of Peurbach (1423—1461), Waltherus (1430—1504), and Regiomontanus (1436—1476). It was given a great impetus by the celebrated German astronomer Copernicus (1473—1545), and has been pursued with increasing zeal to the present time. Copernicus published his masterpiece, *De Revolutionibus Orbium Coelestium,* in 1543, in which he gave to the world the heliocentric theory of the solar system. The system was rejected by Tycho Brahe (1546—1601), who advanced a theory of his own, because he could not observe any parallax in the fixed stars. Tycho was of Norwegian birth, but did much of his astronomical work in Denmark under the patronage of King Frederick. After the death of Frederick he moved to Prague where he was supported the remainder of his life by a liberal pension from Rudolph II. He was an indefatigable and most painstaking observer, and made very important contributions to Astronomy. In his later years Tycho Brahe had Kepler (1571—1630) for his disciple and assistant, and it was by discussing his observations that Kepler was enabled, in less than twenty years after the death of his preceptor, to announce the three laws of planetary motion. It was from these laws that Newton (1642—1727) derived the law of gravitation.

Galileo (1564—1642), an Italian astronomer, a contemporary of Kepler, and a man of greater genius and greater fame, applied the telescope to celestial objects. He discovered four satellites revolving around Jupiter, the rings of Saturn, and spots on the sun. He, like Kepler, was an ardent supporter of the heliocentric theory.

27. Dynamical Astronomy. By Dynamical Astronomy will be meant the connecting of mechanical and physical causes with observed phenomena. Formal Astronomy is so ancient that it is not

possible to go back to its origin; Dynamical Astronomy, on the other hand, did not begin until after the time of Aristotle, and then real advances were made at only very rare intervals.

Archimedes (287—212 B.C.), of Syracuse, is the author of the first sound ideas regarding mechanical laws. He stated correctly the principles of the lever and the meaning of the center of gravity of a body. The form and generality of his treatment were improved by Leonardo da Vinci (1452—1519) in his investigations of statical moments. The whole subject of Statics of a rigid body involves only the application of the proper mathematics to these principles.

It is a remarkable fact that no single important advance was made in the discovery of mechanical laws for nearly two thousand years after Archimedes, or until the time of Stevinus (1548—1620), who was the first, in 1586, to investigate the mechanics of the inclined plane, and of Galileo (1564—1642), who made the first important advance in Kinetics. Thus, the mechanical principles involved in the motions of bodies were not discovered until relatively modern times. The fundamental error in the speculations of most of the investigators was that they supposed that it required a continually acting force to keep a body in motion. This is the opposite of the law of inertia (Newton's first law). This law was discovered by Galileo quite incidentally in the study of the motion of bodies sliding down an inclined plane and out on a horizontal surface. Galileo's division of the mechanical principles into their elements was quite different from that of Newton. He took as his fundamental principle that the change of velocity, or acceleration, is determined by the forces which act upon the body. This contains nearly all of Newton's first two laws. Galileo applied his principles with complete success to the discovery of the laws of falling bodies, and of the motion of projectiles. The value of his discoveries is such that he is justly considered to be the founder of Dynamics. He was the first to employ the pendulum for the measurement of time.

Huyghens (1629—1695), a Dutch mathematician and scientist, published his *Horologium Oscillatorium* in 1675, containing the theory of the determination of the intensity of the earth's gravitation from pendulum experiments, the theory of the center of oscillation, the theory of evolutes, and the theory of the cycloidal pendulum.

Newton (1642—1727) completed the formulation of the fundamental principles of Mechanics, and applied them with unparalleled success in the solution of mechanical and astronomical problems. He

employed Geometry with such skill that his work has scarcely been added to by the use of his methods to the present day

Mathematicians soon turned to the more general and powerful methods of analysis The subject of Analytical Mechanics was founded by Euler (1707—1783) in his work, *Mechanica sive Motus Scientia* (Petersburg, 1736), it was improved by Maclaurin (1698—1746) in his work, *A Complete System of Fluxions* (Edinburgh, 1742), and was completed by Lagrange (1736—1813) in his *Mécanique Analytique* (Paris, 1788)

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CHAPTER II

RECTILINEAR MOTION

28 A great part of the work in Celestial Mechanics consists of the solution of differential equations which, in most problems, are very complicated on account of the number of dependent variables involved. The ordinary Calculus is devoted, in a large part, to the treatment of equations in which there is but one independent variable and one dependent variable, and the step to simultaneous equations in several variables, requiring interpretation in connection with physical problems and mechanical principles, is one which is usually made not without some difficulty The present chapter will be devoted to the mathematical formulation and to the solution of classes of problems in which the mathematical processes are closely related to those which are expounded in the mathematical text-books It will form the bridge leading from methods which are familiar in works on pure Mathematics to those which are characteristic of mechanical and astronomical problems

The examples chosen to illustrate the principles are taken as largely as possible from astronomical problems They are of sufficient interest to justify their insertion, even though they were not needed as a preparation for the more complicated work which will follow They embrace the theory of falling bodies, the velocity of escape, parabolic motion, and the meteoric and contraction theories of the sun's heat

THE MOTION OF FALLING PARTICLES

29 The Differential Equation of Motion Suppose the mass of the particle is m and let s represent the line in which it falls Take the origin O at the surface of the earth and let the positive end of the line be directed upward By the second law of motion the rate of

change of momentum, or the mass times the acceleration, is proportional to the force. Let k^2 represent the factor of proportionality, the numerical value of which will depend upon the units employed. Then, if f represents the force, the differential equation of motion is

$$(1) \qquad m\frac{d\,s}{dt^2} = -k\,f$$

This is also the differential equation of motion for any case in which the resultant of all the forces is constantly in the same straight line and in which the body is not initially projected from that line. A more general treatment will therefore be given than would be required if f were simply the force arising from the earth's attraction for the particle m.

The force f will depend in general upon various things, as the position of m, the time t, and the velocity v. This may be represented by writing equation (1)

$$(2) \qquad m\frac{d\,s}{dt^2} = -k\,\phi\,(s,\,t,\,v),$$

in which $\phi\,(s, t, v)$ simply means that the force may depend upon the quantities contained in the parenthesis. In order to solve (2) two integrations must be performed, and the character of the integrals will depend upon the manner in which ϕ involves s, t, and v. It will be necessary to discuss the different cases separately.

30. Case of Constant Force. This simplest case is nearly realized when particles fall through small distances near the earth's surface under the influence of gravitation. If the second is taken as the unit of time and the foot as the unit of length $k^2 f = mg$, and (1) becomes

$$(3) \qquad \frac{d\,s}{dt^2} = -g$$

This becomes after one integration

$$\frac{ds}{dt} = -gt + c_1,$$

where c_1 is the constant of integration. Let the velocity of the particle at the time $t=0$ be $v = v_0$. Then the last equation becomes at $t = 0$

$$v_0 = c_1\,,$$

whence

$$\frac{ds}{dt} = -gt + v_0$$

The integral of this equation is

$$s = -\frac{gt^2}{2} + v_0 t + c_2$$

Suppose the particle is started at the distance s_0 from the origin at the time $t = 0$, then this equation gives

$$s_0 = c_2 ,$$

whence

$$s = -\frac{gt^2}{2} + v_0 t + s_0 \tag{4}$$

When the initial conditions are given this equation determines the position of the particle at any time, or, it determines the time at which the body has any position by the solution of the quadratic equation in t

If the acceleration were any other positive or negative constant than $-mg$, the equation for the space described would have the same form as (4)

31 Attractive Force Varying Directly as the Distance Another simple case is that in which the force varies directly as the distance from the origin Suppose it is always attractive toward the origin This has been found by experiment to be very nearly the law of tension of an elastic string within certain limits of stretching Then the velocity will decrease when the particle is on the positive side of the origin, therefore for these positions of the particle the acceleration must be taken with the negative sign, and the differential equation for positive values of s is

$$m\frac{d^2s}{dt^2} = -k^2 s \tag{5}$$

For positions of the particle in the negative direction from the origin the velocity increases with the time, and therefore the acceleration must be positive The right member of equation (5) must be taken with such a sign that it will be positive Since s is negative in the region under consideration the negative sign must be prefixed, and the equation remains as before Equation (5) is, therefore, the general differential equation of motion for a body subject to an attractive force varying directly as the distance

After multiplying equation (5) by $\frac{ds}{dt}$ the left member is the derivative

of $\frac{m}{2}\left(\frac{ds}{dt}\right)^2$ with respect to t, as is easily verified, and the right member is the derivative of $-\frac{1}{2}k^2s^2$ Hence integrating once, (5) becomes

$$\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = -\tfrac{1}{2}k^2s^2 + c_1$$

If $s = s_0$ and $\frac{ds}{dt} = 0$, at the time $t = 0$, then this equation becomes

$$\left(\frac{ds}{dt}\right) = \frac{k}{m}(s_0^2 - s\ ),$$

which may be written, as is customary in separating the variables,

$$\frac{ds}{\sqrt{s_0 \ - s^2}} = \pm\frac{kdt}{\sqrt{m}}$$

The integral of this equation is

$$\sin^{-1}\frac{s}{s_0} = \pm\frac{kt}{\sqrt{m}} + c$$

It is found from the initial conditions that $c_1 = \frac{\pi}{2}$, whence

(6) $$\sin^{-1}\frac{s}{s_0} = \pm\frac{kt}{\sqrt{m}} + \frac{\pi}{2}$$

Taking the sine of both members, this equation becomes

(7) $$s = s_0 \sin\left(\pm\frac{kt}{\sqrt{m}} + \frac{\pi}{2}\right) = s_0 \cos\left(\frac{kt}{\sqrt{m}}\right)$$

From this equation it is seen that the motion is oscillatory and symmetrical with respect to the origin, with a period of $\frac{2\pi\sqrt{m}}{k}$ In the practical example of the tension of a string the law does not hold while the body is near the origin, hence (7) must be regarded as being the solution of an ideal problem The conditions would be more nearly realized if the body were a ring, sliding on a wire without friction, the point of attachment of the elastic string being at a distance from the wire greater than its normal length

III PROBLEMS

1 A particle is started with an initial velocity of 20 meters per sec and is subject to an acceleration of 20 meters per sec What will be its velocity at the end of 4 secs, and how far will it have moved?

$$\textit{Ans} \quad \begin{cases} v=100 \text{ meters per sec} \\ s=240 \text{ meters} \end{cases}$$

2 A particle starting with an initial velocity of 10 meters per sec and moving with a constant acceleration describes 2050 meters in 5 secs What is the acceleration?

Ans $a=160$ meters per sec

3 A particle is moving with an acceleration of 5 meters per sec Through what space must it move in order that its velocity may be increased from 10 meters per sec to 20 meters per sec?

Ans 30 meters

4 A particle starting with an initial velocity of 10 meters per sec and moving under an acceleration of 20 meters per sec described a space of 420 meters What time was required?

Ans $t=6$ secs

5 Show that, if a particle moves subject to an attractive force varying directly as the distance, the time of falling from any point to the origin is independent of the distance of the point

6 Suppose a particle moves subject to an attractive force varying directly as the distance, and that the acceleration at a distance of 1 meter is 1 meter a sec If the particle starts from rest how long will it take it to fall from a distance of 20 meters to 10 meters?

Ans 1 0472 secs

7 Suppose the force is repulsive from the origin and that it varies directly as the distance, and show that, if $v=0$ and $s=s_0$ when $t=0$,

$$\log\left(\frac{s+\sqrt{s^2-s_0^2}}{s_0}\right)=\frac{k}{\sqrt{m}}t,$$

whence, letting $\frac{k}{\sqrt{m}}=K$,

$$s=\frac{s_0}{2}(e^{Kt}+e^{-Kt})$$

Observe that equation (7) may be written in the similar form

$$s=\frac{s_0}{2}(e^{\sqrt{-1}Kt}+e^{-\sqrt{-1}Kt})$$

32 Solution by the General Method Equation (5) and the corresponding one for a repulsive force are linear in s, and may be solved in the following manner, which is efficient for all linear homogeneous equations with constant coefficients* Consider the double form

$$(8) \qquad \begin{cases} \dfrac{d^2 s}{dt^2} + k\, s = 0, \\[2ex] \dfrac{d^2 s}{dt^2} - k\, s = 0 \end{cases}$$

Assume $s = e^{\lambda t}$ and substitute in the differential equations, whence

$$\begin{cases} \lambda^2 e^{\lambda t} + k\, e^{\lambda t} = 0, \\ \lambda\, e^{\lambda t} - k\, e^{\lambda t} = 0 \end{cases}$$

$e^{\lambda t}$ can vanish or become infinity only for $t = \mp \infty$, and these values of the variable will be excluded, therefore this factor may be divided out and there results

$$(9) \qquad \begin{cases} \lambda^2 + k^2 = 0, \\ \lambda^2 - k^2 = 0 \end{cases}$$

Let the roots of the first equation be λ_1 and λ_2, then the first equation of (8) is verified by both of the particular solutions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ The general solution is the sum of these two particular solutions each multiplied by an arbitrary constant Precisely similar results hold for the second equation of (8) Putting in the value of the roots, the respective general solutions are

$$(10) \qquad \begin{cases} s = c_1 e^{\sqrt{-1}kt} + c_2 e^{-\sqrt{-1}kt}, \\ s = c_1' e^{kt} + c' e^{-kt}, \end{cases}$$

where c_1, c_2, c_1', and c_2' are the constants of integration Suppose $s = s_0$, and $\dfrac{ds}{dt} = 0$ when $t = 0$, therefore

$$\begin{cases} s_0 = c_1 + c_2, \\ s_0 = c_1' + c_2' \end{cases}$$

The derivatives of (10) are

$$\begin{cases} \dfrac{ds}{dt} = c_1 \sqrt{-1}\, k e^{\sqrt{-1}kt} - c_2 \sqrt{-1}\, k e^{-\sqrt{-1}kt}, \\[2ex] \dfrac{ds}{dt} = c_1' k e^{kt} - c_2' k e^{-kt} \end{cases}$$

Substituting $t = 0$ and $\dfrac{ds}{dt} = 0$, it follows that

$$\begin{cases} c_1 \sqrt{-1}\, k - c_2 \sqrt{-1}\, k = 0, \\ c_1' k - c_2 k = 0 \\ \quad c_1 = c_2, \\ \quad c_1' = c_2' \end{cases}$$

* See Johnson's *Differential Equations*, Art 95

Then the general solution becomes

$$(11)\qquad \begin{cases} s = \dfrac{s_0}{2}\left(e^{\sqrt{-1}kt} + e^{-\sqrt{-1}kt}\right), \\ s = \dfrac{s_0}{2}\left(e^{kt} + e^{-kt}\right) \end{cases}$$

This shows the relation between the two problems much more clearly than the other method of solving

33 Attractive Force Varying Inversely as the Square of the Distance For positions in the positive direction from the origin the velocity decreases as the time increases, therefore in this region the acceleration is negative Similarly, on the negative side of the origin the acceleration is positive Since $\frac{k^2}{s^2}$ is always positive the right member has different signs in the two cases For simplicity suppose the mass of the attracted particle is unity Then the differential equation of motion for all the positions of the particle in the positive direction from the origin is

$$(12)\qquad \frac{d^2s}{dt^2} = -\frac{k^2}{s^2}$$

Multiplying both members by $2\frac{ds}{dt}$ and integrating, it is found that

$$(13)\qquad \left(\frac{ds}{dt}\right)^2 = \frac{2k^2}{s} + c_1$$

Suppose $v = v_0$ and $s = s_0$, when $t = 0$, then

$$c_1 = v_0^2 - \frac{2k^2}{s_0}$$

Substituting in (13),

$$\frac{ds}{dt} = \pm\sqrt{\frac{2k^2}{s} + v_0^2 - \frac{2k^2}{s_0}}$$

If $v_0^2 - \frac{2k^2}{s_0} < 0$ there will be some finite distance, s, at which $\frac{ds}{dt}$ will vanish, if the direction of motion of the particle is such that it reaches that point it will turn there and move in the other direction If $v_0^2 - \frac{2k^2}{s_0} = 0$, $\frac{ds}{dt}$ will vanish at $s = \infty$, and if the particle moves from the origin it will approach infinity as the velocity approaches zero If $v_0^2 - \frac{2k^2}{s_0} > 0$, $\frac{ds}{dt}$ never vanishes, and if the particle moves from the origin it will approach infinity with a finite velocity

Suppose the first condition is fulfilled and that $\frac{ds}{dt} = 0$ when $s = s_1$ Then equation (13) gives

$$(14) \qquad \frac{ds}{dt} = \pm \frac{\sqrt{2}k}{\sqrt{s_1}} \sqrt{\frac{s_1 - s}{s}}$$

The positive or negative sign is to be taken according as the particle is receding from, or approaching, the origin The equation may be written in case the particle is approaching the origin

$$\frac{-sds}{\sqrt{s_1 s - s}} = \frac{\sqrt{2}\,kdt}{\sqrt{s_1}},$$

or again

$$\frac{1}{2}\frac{(s_1 - 2s)\,ds}{\sqrt{s_1 s - s^2}} - \frac{s_1}{2}\frac{ds}{\sqrt{s_1 s - s}} = \frac{\sqrt{2}\,kdt}{\sqrt{s_1}}$$

The integral is

$$\sqrt{s_1 s - s^2} - \frac{s_1}{2} \operatorname{vers}^{-1}\left(\frac{2s}{s_1}\right) = \frac{\sqrt{2}\,kt}{\sqrt{s_1}} + c_2$$

Since $s = s_0$ when $t = 0$ it follows that

$$c_2 = \sqrt{s_0 s_1 - s_0} - \frac{s_1}{2} \operatorname{vers}^{-1} \frac{2s_0}{s_1},$$

whence

$$(15) \quad k\sqrt{\frac{2}{s_1}}\,t = \sqrt{s_1 s - s} - \frac{s_1}{2}\operatorname{vers}^{-1}\left(\frac{2s}{s_1}\right) - \sqrt{s_0 s_1 - s_0^2} + \frac{s_1}{2}\operatorname{vers}^{-1}\left(\frac{2s_0}{s_1}\right)$$

This equation determines the time at which the particle has any position at the right of the origin and at a distance from it less than s_1 For values of s greater than s_1, and for all negative values of s, the first term on the right becomes imaginary That means that the equation does not hold for these values of the variables, this was indeed certain because the differential equations (13) and (14) were both true only in the region

$$0 < s \leqq s_1$$

Putting s equal to zero in (15) the expression is obtained for the time it takes the particle to fall from the point s_1 to the center of force It is

$$(16) \qquad T = -\frac{1}{k}\sqrt{\frac{s_0 s_1 - s_1 s_0^2}{2}} + \frac{s_1^{\frac{3}{2}}}{k\,2^{\frac{1}{2}}} \operatorname{vers}^{-1}\left(\frac{2s_0}{s_1}\right)$$

From equation (14) s_1 can easily be expressed in terms of s_0 and v_0

34 The Height of Projection Suppose $v_0 = 0$, then s_1 s_0 If the particle starting from rest at s_0 falls to s_2 and acquires the velocity v_2, it will, if projected from s_2 in the direction of s_0 with

the velocity v_2, recede exactly to s_0, for, all the circumstances of motion will be the reverse of those in the motion from s_0 to s_2

The direct proof from the equations is also very simple The square of the velocity acquired in falling from s_0 to s_2 is found from (14) to be in this case

$$v_2^2 = 2k^2\left(\frac{1}{s_2} - \frac{1}{s_0}\right)$$

Now consider the case where s_2 is the starting point, with the initial velocity v_2 The constant c_1 has the value

$$c_1 = v_2^2 - \frac{2k^2}{s_2},$$

and the equation for the square of the velocity at any point is

$$\left(\frac{ds}{dt}\right)^2 = 2k^2\left(\frac{1}{s} - \frac{1}{s_2}\right) + v_2^2$$

The particle will recede to the point where the velocity becomes zero Substituting the value of v_2^2, given above, in this equation and imposing the condition that $\left(\frac{ds}{dt}\right)^2$ shall be zero, there results for the corresponding value of s

$$s = s_0 \qquad \text{Q. E D}$$

35 The Velocity from Infinity When the particle starts from s_0 with zero velocity, equation (14) becomes

$$(17) \qquad \left(\frac{ds}{dt}\right)^2 = 2k^2\left(\frac{1}{s} - \frac{1}{s_0}\right)$$

If the particle falls from an infinite distance, s_0 is infinite and (17) reduces to

$$(18) \qquad \left(\frac{ds}{dt}\right)^2 = \frac{2k^2}{s}$$

From the investigations of Art 34 it follows that, if the particle be projected from any point s in the positive direction with the velocity defined by (18), it will recede to infinity The law of attraction in deriving (18) was Newton's law of gravitation, therefore this equation may be used for the computation of the velocity which a particle starting from infinity would acquire in falling to the surfaces of the various planets, satellites, and the sun Then, if the particle were projected from the surfaces of the various members of the solar system with these respective velocities, it would recede to an infinite distance and be lost to the system

36 Application to the Escape of Atmospheres The kinetic theory of gases is that the volumes and pressures of gases

are maintained by the mutual impacts of the individual molecules, which are, on the average, in a state of very rapid motion The rarity of the earth's atmosphere and the fact that the pressure is about fifteen pounds to the square inch, serve to give some idea of the high speed with which the molecules move and of the great frequency of their impacts The different molecules do not all move with the same speed in any given gas under fixed conditions, but the number of those which have a rate of motion different from the mean of the squares becomes very rapidly less as the differences increase Theoretically, in all gases the range of the values of the velocities is from zero to infinity, although the extreme cases occur at infinitely rare intervals compared to the others Under constant pressure the velocities are directly proportional to the square root of the temperature, and inversely proportional to the square root of the molecular weight

Since in all gases all velocities exist, some of the molecules of the gaseous envelopes of the heavenly bodies must be moving with velocities greater than the *velocity from infinity*, as defined in Art 35 If the molecules are near the upper limits of the atmosphere, and start away from the body to which they belong, they may miss collisions with other molecules and escape never to return* It is as certain, therefore, as that the kinetic theory of gases is true, that the atmospheres of all the celestial bodies are being depleted by this process, but in most cases it is an excessively slow one, and is compensated, to some extent at least, by the accretion of meteoric matter and atmospheric molecules from other bodies In the upper regions of the gaseous envelopes, from which alone the molecules escape, the temperatures are low, at least for planetary bodies like the earth, and high velocities are of very rare occurrence If the mean square velocity were as great as, or exceeded, the velocity from infinity the depletion would be a relatively rapid process In any case the elements and compounds with low molecular weights would be lost most rapidly, and thus certain ones might escape and others be retained

The manner in which the velocity from infinity with respect to the surface of an attracting sphere varies with its mass and radius will now be investigated The mass is proportional to the product of the density and cube of the radius Then k^2, which is the attraction at unit distance, varies directly as the mass, and therefore directly as the product of the density and cube of the radius Hence it follows from

* This theory is due to Dr Johnstone Stoney, *Trans Royal Dublin Soc* 1892

(18) that the velocity from infinity at the surface of the planet varies as the product of the radius and the square root of the density The densities and the radii of the members of the solar system are usually expressed in terms of the density and the radius of the earth, hence the velocity from infinity may be easily computed for all of them when it has been determined for the earth

Let R represent the radius of the earth, and g the acceleration at its surface Then from the definition of k^2, it follows that

$$g = \frac{k^2}{R^2} \tag{19}$$

Substituting the value of k^2 determined from this equation into (18), it becomes

$$\left(\frac{ds}{dt}\right) = \frac{2gR^2}{s}$$

Let $V = \frac{ds}{dt}$ when $s = R$, whence

$$V^2 = 2gR$$

Let a second be taken as the unit of time, and a meter as the unit of length Then $R = 6,371,000$, and $g = 9\ 8094$ * Substituting in the last equation and carrying out the computation, it is found that $V = 11,180$ meters, or about 6 95 miles, per sec Taking the values of the relative densities and radii of the planets as given in the appendix to Young's *General Astronomy*, the following table was computed

Body	*Velocity from infinity*						
Earth	11,180	meters,	or	6 95	miles,	per	sec
Moon	2,383	,,	,,	1 48	,,	,,	,,
Sun	611,543	,,	,,	380 00	,,	,,	,,
Mercury	3,938	,,	,,	2 45	,,	,,	,,
Venus	10,250	,,	,,	6 37	,,	,,	,,
Mars	5,030	,,	,,	3 13	,,	,,	,,
Jupiter	59,810	,,	,,	37 16	,,	,,	,,
Saturn	36,964	,,	,,	22 97	,,	,,	,,
Uranus	21,133	,,	,,	13 13	,,	,,	,,
Neptune	21,950	,,	,,	13 64	,,	,,	,,

The velocity from infinity decreases as the distance from the center of a planet increases, and the necessary velocity of projection in order that a molecule may escape decreases correspondingly, and this is still further reduced by the centrifugal acceleration of the planet's rotation

* *Annuaire du Bureau des Long* 1900 g is given for the lat of Paris, 48° 50′

The question arises whether, under the conditions prevailing at the surfaces of the various bodies enumerated, the average molecular velocities of the atmospheric elements do not equal or surpass the corresponding velocity from infinity

It is possible to find experimentally the pressure exerted by a given mass of gas upon a given surface at a given temperature, from which the mean square velocity may be computed In Risteen's *Molecules and the Molecular Theory*, p 39, it is stated that the square root of the mean square velocity of hydrogen molecules at the temperature 0 Centigrade under atmospheric pressure is 5,571 feet per second Under the same conditions the velocity of oxygen and nitrogen molecules would be about one-fourth as much

The surface temperatures of the inferior planets are certainly much greater than zero degrees Centigrade in the parts where they receive the rays of the sun most directly, even if all the heat which may ever have been received from their interiors is neglected It is certain from the geological evidences of igneous action upon the earth that in the remote past they were at a much higher temperature, and it is probable that the superior planets have not yet cooled down to the solid state There is evidence that the most refractory substances have been in a molten state at some time, which implies that they have had a temperature of 3000 or 4000 degrees Centigrade * Therefore the square root of the mean square velocity must have been much greater than the amount given above, and it probably continued much greater for a long period of time Comparing these results with the table on velocities from infinity it is seen that the moon and inferior planets, according to this theory, could not possibly have retained free hydrogen and other elements of very small molecular weights, such as helium, in their envelopes, in the case of the moon, Mercury, and Mars, the escape of heavier molecules as nitrogen and oxygen must have been frequent This is especially probable when the heated atmospheres extended out to great distances The superior planets, and much more so the sun, could have retained all of the familiar terrestrial elements, and from this theory it should be expected that they would be surrounded with extensive gaseous envelopes

The observed facts are that the moon has no appreciable atmosphere whatever, Mercury an extremely rare one, if any at all, Mars, one much less dense than that of the earth, Venus, one perhaps

* *A Group of Hypotheses Bearing on Climatic Changes*, T C Chamberlin, *Geol Jour Oct —Nov* 1897

somewhat more dense than that of the earth, on the other hand the superior planets are probably surrounded by extensive gaseous envelopes Their positions and physical conditions are such that it is not possible to determine with certainty how dense their atmospheres really are

37 The Force Proportional to the Velocity When a particle moves in a resisting medium the forces to which it is subject depend upon its velocity Experiments have shown that in the earth's atmosphere when the velocity is less than 3 meters per second the resistance varies nearly as the first power of the velocity, for velocities from 3 to 300 meters per second it varies nearly as the second power of the velocity, and for velocities about 400 meters per second, nearly as the third power of the velocity

(*a*) Consider first the case where the resistance varies as the first power of the velocity, and suppose the motion is on the earth's surface in a horizontal direction with no force acting except that arising from the resistance Then the differential equation for motion in the positive direction is

$$\frac{d^2s}{dt^2} + k^2\frac{ds}{dt} = 0 \tag{20}$$

This is linear in the dependent variable s, and the general method of solving linear equations may be applied

Assume the particular solution

$$s = e^{\lambda t}$$

Substitute in (20) and divide by $e^{\lambda t}$, then

$$\lambda^2 + k^2\lambda = 0$$

The roots of this equation are

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = -k^2, \end{cases}$$

and the general solution is

$$\begin{cases} s = c_1 + c_2 e^{-k^2 t}, \\ \dfrac{ds}{dt} = -c_2 k \; e^{-k^2 t} \end{cases} \tag{21}$$

Suppose $\frac{ds}{dt} = v_0$ and $s = s_0$ when $t = 0$ Then the constants c_1 and c_2 may be determined in terms of v_0 and s_0

(*b*) Consider the case where the resistance varies as the first power of the velocity and suppose motion is in the vertical line Take the

positive end of the axis upward Then the differential equation for motion upward is

$$\frac{d^2s}{dt^2}+k^2\frac{ds}{dt}=-g, \tag{22}$$

and when the body is descending

$$\frac{d^2s}{dt^2}-k^2\frac{ds}{dt}=-g \tag{23}$$

These equations will be solved by the important method of *The Variation of Parameters** This is one of the most effective processes of treating the differential equations which arise in astronomical problems, and it will be used in equations of a much more difficult type in the chapter on Perturbations

Equations (22) and (23) would be linear and homogeneous if their right members were zero This case has just been solved (Art 37), and the results obtained will be made the first step in solving the more general case where the right member is not zero The method consists in this The constants of integration c_1 and c_2 in the first equation of (21) will now be taken as variables depending upon the time, and they will be determined in such a manner that equation (22) shall be fulfilled There are the two functions c_1 and c_2 available for use in satisfying the one equation (22), therefore one arbitrary condition may be imposed upon them It will be chosen so that it will simplify matters as much as possible

It remains to substitute (21) in (22) considering c_1 and c_2 as functions of t to be determined Differentiating once, (21) becomes

$$\frac{ds}{dt}=-k^2c_2e^{-k^2t}+\frac{dc_1}{dt}+e^{-k^2t}\frac{dc_2}{dt} \tag{24}$$

The condition will now be imposed that

$$\frac{dc_1}{dt}+e^{-k^2t}\frac{dc_2}{dt}=0 \tag{25}$$

Differentiating (24), it follows that

$$\frac{d^2s}{dt^2}=k^4c_2e^{-k^2t}-k^2e^{-k^2t}\frac{dc_2}{dt} \tag{26}$$

Substituting (24) and (26) in (22), the second condition upon the functions c_1 and c_2 is found to be

$$k^2\frac{dc_1}{dt}=-g$$

* See Johnson's *Differential Equations*, p 84

Integrating

$$(27) \qquad c_1 = -\frac{gt}{k^2} + c_1'$$

Substituting (27) in (25) and solving for c_2, it is found that

$$(28) \qquad c_2 = \frac{g}{k^4} e^{k^2 t} + c_2'$$

Substituting the values of c_1 and c_2 defined by (27) and (28) respectively in the first equation of (21), the general solution of (22) is found to be

$$(29) \qquad s = -\frac{gt}{k^2} + c_1' + \frac{g}{k^4} + c_2' e^{-k^2 t}$$

This is the general solution since it contains two arbitrary constants c_1' and c_2', and when substituted in (22) identically fulfills it. It will also be observed that if a function of the time had appeared as the right member of (22) instead of the constant g, the complete solution could have been found by the same method. If v and s are given at $t=0$ the constants c_1' and c_2' may be determined by the usual method.

In a similar manner the solution of (23) is found to be

$$(30) \qquad s = \frac{gt}{k^2} + c_1'' + \frac{g}{k^4} + c_2'' e^{k^2 t}$$

The velocities for ascent and descent are respectively

$$(31) \qquad \begin{cases} \left(\dfrac{ds}{dt}\right)_a = \dfrac{-g}{k^2} - c_2' k^2 e^{-k^2 t}, \\ \left(\dfrac{ds}{dt}\right)_d = \dfrac{g}{k^2} + c_2'' k^2 e^{k^2 t} \end{cases}$$

Suppose the particle is projected from the origin upward with the velocity v_0. That is, at the origin of time $t=0$, $\dfrac{ds}{dt} = v_0$, and $s=0$. From (29) and the first of (31) it follows that

$$\begin{cases} c_1' + c_2' + \dfrac{g}{k^4} = 0, \\ c_2' + \dfrac{g}{k^4} + \dfrac{v_0}{k^2} = 0, \end{cases}$$

whence

$$\begin{cases} c_1' = \dfrac{v_0}{k^2}, \\ c_2' = -\dfrac{v_0}{k^2} - \dfrac{g}{k^4} \end{cases}$$

Substituting this value of c_2' in (31), putting $\left(\dfrac{ds}{dt}\right)_a$ equal to zero, and

solving, it is found that the time at which the particle will have reached its highest point is given by the equation

$$e^{k^2 T} = 1 + \frac{k^2 v_0}{g}$$

Substituting in (29), it is found that the greatest height to which the particle will rise is given by

$$S = -\frac{g}{k^4} \log\left(1 + \frac{k^2 v_0}{g}\right) + \frac{v_0}{k^2}$$

38 The Force Proportional to the Square of the Velocity At the velocity of a strong wind, or of a body falling any considerable distance, or of a ball thrown, the resistance varies very nearly as the square of the velocity An investigation will now be made of the character of the motion of a particle when projected upward against gravity, and subject to a resistance from the atmosphere varying as the square of the velocity For simplicity in writing, the acceleration due to resistance at unit velocity will be taken as $k\,g$ Then the differential equation for a unit particle is

$$(32) \qquad \frac{d^2 s}{dt^2} = -g - k^2 g \left(\frac{ds}{dt}\right)$$

This equation may be written

$$\frac{\frac{d}{dt}\left(k\frac{ds}{dt}\right)}{1 + k\left(\frac{ds}{dt}\right)^2} = -kg,$$

of which the integral is

$$(33) \qquad \tan^{-1}\left(k\frac{ds}{dt}\right) = -kgt + c_1$$

If $\frac{ds}{dt} = v_0$ and $s_0 = 0$, when $t = 0$, then

$$c_1 = \tan^{-1}(kv_0)$$

Substituting in (33) and taking the tangent of both members, it follows that

$$(34) \qquad \frac{ds}{dt} = \frac{1}{k}\,\frac{v_0 k - \tan(kgt)}{1 + v_0 k \tan(kgt)}$$

This equation expresses the velocity in terms of the time Multiplying both numerator and denominator of the right member of (34) by $\cos(kgt)$ the numerator becomes the derivative of the denominator with respect to the time Then integrating

$$(35) \qquad s = \frac{1}{k^2 g} \log\left[v_0 k \sin(kgt) + \cos(kgt)\right] + c_2$$

It follows from the initial conditions that $c_2 = 0$ This equation expresses the distance passed over in terms of the time

The equations may be treated so that the velocity will be expressed in terms of the distance Equation (32) may be written

$$\frac{\frac{d}{dt}\left\{k^2\left(\frac{ds}{dt}\right)^2\right\}}{1 + k^2\left(\frac{ds}{dt}\right)^2} = -2gk^2\frac{ds}{dt},$$

of which the integral is

$$\log\left\{1 + k^2\left(\frac{ds}{dt}\right)^2\right\} = -2gk^2 s + c_1'$$

From the initial conditions it follows that

$$c_1' = \log(1 + k^2 v_0^2)$$

Therefore

$$\left(\frac{ds}{dt}\right)^2 = \frac{1}{k^2}(1 + k^2 v_0^2)\, e^{-2gk^2 s} - \frac{1}{k^2} \tag{36}$$

The maximum height is reached when the velocity becomes zero, and is found from (36) to be

$$S = \frac{1}{2gk^2}\log(1 + k^2 v_0^2)$$

Putting $\frac{ds}{dt} = 0$ in (34), the time of reaching the highest point is given by

$$T = \frac{1}{kg}\tan^{-1}(v_0 k)$$

When the particle falls the resistance acts in the opposite direction and (32) has the sign of the last term changed This may be accomplished by writing $k\sqrt{-1}$ instead of k, and if this change be made throughout the solution the results will be valid Of course the results should be written in the exponential form, instead of the trigonometric as they were in (34), in order to avoid the appearance of imaginary expressions If the initial velocity is zero, $v_0 = 0$ and the equations corresponding to (34), (35), and (36) are respectively

$$\left\{\begin{aligned} \frac{ds}{dt} &= -\frac{1}{k}\frac{e^{kgt} - e^{-kgt}}{e^{kgt} + e^{-kgt}}, \\ e^{-gk^2 s} &= \frac{e^{kgt} + e^{-kgt}}{2}, \\ \left(\frac{ds}{dt}\right)^2 &= \frac{1}{k^2}(1 - e^{2gk^2 s}) \end{aligned}\right. \tag{37}$$

IV PROBLEMS

1 Solve equation (20) by direct integration and show that the same result is found as by the method for linear equations

2 Let $s = s' \mp \frac{g}{k^2} t$ in equations (22) and (23) respectively, integrate directly and show that the result is the same as that found by the variation of parameters

3 Find equations (37) by direct integration of the differential equations

4 Suppose a particle starts from rest and moves subject to a repulsive force varying inversely as the square of the distance, find the velocity and time elapsed in terms of the space described

$$Ans \quad \begin{cases} v^2 = 2k^2\left(\dfrac{1}{s_0} - \dfrac{1}{s}\right), \\ k\sqrt{\dfrac{2}{s_0}}\, t = \sqrt{s^2 - s_0 s} + \dfrac{s_0}{2}\log\left(\dfrac{\sqrt{s^2 - s_0 s} + s - \dfrac{s_0}{2}}{\dfrac{s_0}{2}}\right) \end{cases}$$

5 What is the velocity from infinity with respect to the sun at the earth's distance?

Ans 41,850 meters, or 26 miles, per sec

6 Suppose a particle moves subject to an attractive force varying directly as the distance, and to another proportional to the velocity, solve the differential equation by the general method for linear equations

Ans Let k^2 be the factor of proportionality for the velocity and l^2 for the distance Then the solutions are

$$s = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

where

$$\begin{cases} \lambda_1 = \dfrac{-k^2 + \sqrt{k^4 - 4l^2}}{2}, \\ \lambda_2 = \dfrac{-k^2 - \sqrt{k^4 - 4l^2}}{2} \end{cases}$$

7 Suppose that in addition to the forces of problem 6 there is a force $\mu e^{\nu t}$, discuss the motion by the method of the variation of parameters

8 Develop the method of the variation of parameters for a linear differential equation of the third order

39 Parabolic Motion. There is a class of problems involving for their solution mathematical processes which are similar to those employed thus far in this chapter, although the motion is not in a straight line On account of the similarity in the analysis a short account of these trajectories will be inserted here

Suppose the particle is subject to a constant acceleration downward, it is required to discuss the curve described when the particle is projected in any manner The orbit will be in a plane which will be taken as the xy plane Let the y-axis be vertical with the positive end directed upward Then the differential equations of motion are

$$(38)\qquad \begin{cases} \dfrac{d^2x}{dt^2}=0, \\ \dfrac{d^2y}{dt^2}=-g \end{cases}$$

Since these equations are independent, they may be integrated separately, and give

$$\begin{cases} x=a_1t+a_2, \\ y=-\dfrac{gt^2}{2}+b_1t+b_2 \end{cases}$$

Suppose $x=y=0$, $\dfrac{dx}{dt}=v_0\cos\alpha$, $\dfrac{dy}{dt}=v_0\sin\alpha$ when $t=0$, where α is the angle between the line of initial projection and the plane of the

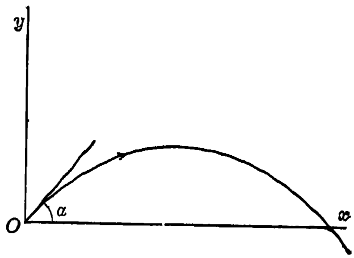

Fig 6

horizon, and v_0 is the speed of the projection Then the constants of integration are found to be

$$a_1=v_0\cos\alpha,\qquad a_2=0,$$
$$b_1=v_0\sin\alpha,\qquad b_2=0,$$

and therefore

$$(39)\qquad \begin{cases} x=v_0\cos\alpha\ t, \\ y=-\dfrac{gt^2}{2}+v_0\sin\alpha\ t \end{cases}$$

The equation of the curve described is found by eliminating t between these two equations, and is

$$(40) \qquad y = x \tan \alpha - \frac{g \sec \ \alpha}{2v_0} \ x$$

This is the equation of a parabola whose axis is vertical and vertex upward It may be written

$$\left(x - \frac{v_0^2}{g} \sin \alpha \cos \alpha\right)^2 = -\frac{2v_0^2}{g \sec \ \alpha}\left(y - \frac{v_0^2 \sin^2 \alpha}{2g}\right)$$

The equation of a parabola with its vertex as the origin has the form

$$x^2 = 2py,$$

where $2p$ is the parameter Comparing with the last equation the coordinates of the vertex, or highest point, are seen to be

$$\begin{cases} x = \dfrac{v_0^2}{g} \sin \alpha \cos \alpha, \\[2ex] \bar{y} = \dfrac{v_0^2 \sin \ \alpha}{2g} \end{cases}$$

The distance of the directrix from the vertex is one-fourth of the parameter, therefore the equation of the directrix is

$$y = \bar{y} - \frac{p}{2} = \frac{v_0^2 \sin^2 \alpha}{2g} + \frac{v_0^2 \cos \ \alpha}{2g} = \frac{v_0^2}{2g}$$

The square of the velocity is found to be

$$v \ = \left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)^2 = v_0 \ - 2gy$$

To find the place where the particle will strike the horizontal plane put $y = 0$ in (40) The solutions for x are $x = 0$ and

$$x = \frac{2v_0^2}{g} \sin \alpha \cos \alpha = \frac{v_0}{g} \sin 2\alpha$$

From this it follows that the range will be greatest for a given initial velocity if $\alpha = 45$ From (39) the horizontal velocity is $v_0 \cos \alpha$, hence the time of flight is $\dfrac{2v_0}{g} \sin \alpha$ Therefore, if the other initial conditions are kept fixed, the whole time of flight varies directly as the sine of the angle of elevation

The angle of elevation to attain a given range is found by solving

$$x = a = \frac{v_0^2}{g} \sin 2\alpha$$

for α If $a > \frac{v_0^2}{g}$ there is no solution If $a < \frac{v_0^2}{2}$ there are two solutions for α differing from the value for maximum range ($\alpha = 45°$) by equal amounts

If the variation in gravity at different heights above the earth's surface, the curvature of the earth, and the resistance of the air be neglected, the investigation above applies to projectiles near the earth's surface For bodies of great density the results given by this theory are tolerably accurate for short ranges When the acceleration is taken toward the center of the earth, and gravity is supposed to vary inversely as the square of the distance, the trajectory is an ellipse with the center of the earth as one of the foci This will be proved in the next chapter

V PROBLEMS

1 Prove that, if the accelerations parallel to the x and y axes are any constant quantities, the path described by the particle is a parabola for general initial conditions

2 Find the direction of the major axis obtained in problem 1 in terms of the constant components of acceleration

3 Under the assumptions of Art 39 find the range on a line making an angle β with the x axis

4 Show that the direction of projection for the greatest range on a given line passing through the point of projection is in a line bisecting the angle between the given line and the y-axis

5 Show that the locus of the highest points of the parabola as α takes all values is an ellipse whose major axis is $\frac{v_0^2}{g}$, and minor axis, $\frac{v_0^2}{2g}$

The Heat of the Sun

40 Work and Energy When a force moves a particle against any resistance it is said to do *work* The amount of the work is proportional to the product of the resistance and the distance through which the particle is moved In the case of a free particle the resistance comes entirely from the inertia of the mass, if there is friction this is also resistance

Energy is the power of doing work If a given amount of work is done upon a particle free to move, the particle acquires a motion that will enable it to do exactly the same amount of work The energy of motion is called *kinetic energy* If the particle is retarded by friction part of the original work expended will be used in overcoming the friction, and the particle will be capable of doing only as much work as has been done in giving it motion Until about fifty years ago it was supposed that work done in overcoming friction was lost, or that at least a part of it was In other words, it was believed that the total amount of energy in an isolated system would continually decrease It was observed, however, that friction generates heat, sound, light, and electricity, depending upon the circumstances, and that these manifestations of energy were of the same nature as the original, but in a different form The next step was to prove that these modified forms of energy were in every case quantitatively equivalent to the waste of that originally considered The brilliant experiments of Joule made in the middle of the nineteenth century have established with great certainty the fact that the total amount of energy remains the same whatever changes it may undergo This principle, known as the *conservation of energy*, when stated as holding throughout the universe, is probably the most far-reaching generalization that has been made in the natural sciences in a hundred years *

41 Computation of Work The amount of work done by a Newtonian force in moving a free particle any distance will now be computed Let m equal the mass of the particle moved, and k a constant depending upon the units employed Then

$$m\frac{d^2s}{dt} = -\frac{k\,m}{s^2} \qquad (41)$$

* Herbert Spencer regards the principle as being axiomatic, and states his views in regard to it in his *First Principles*, part II chap VI

The right member is the force to which the particle is subject By Newton's third law it is numerically equal to the reaction, or the resistance due to inertia Then the work done in moving the particle through the element of distance ds is

$$m\frac{d^2s}{dt^2}ds = -\frac{k^2m}{s^2}ds$$

The work done in moving the particle through the interval from s_0 to s_1 is found by taking the definite integral between the limits s_0 and s_1 Performing the integrations and substituting the limits, it is found that

$$\frac{m}{2}\left(\frac{ds_1}{dt}\right)^2 - \frac{m}{2}\left(\frac{ds_0}{dt}\right)^2 = k^2m\left(\frac{1}{s_1}-\frac{1}{s_0}\right)$$

Suppose the initial velocity is zero , then the kinetic energy equals the work W done upon the particle, and

$$(42)\qquad W = \frac{m}{2}\left(\frac{ds_1}{dt}\right)^2 = k^2m\left(\frac{1}{s_1}-\frac{1}{s_0}\right)$$

The particle had no kinetic energy on the start, and therefore the power of doing work equals the product of one half the mass and the square of the velocity If the particle falls from infinity, s_0 becomes infinite, and the formula for the kinetic energy is

$$(43)\qquad \frac{m}{2}\left(\frac{ds_1}{dt}\right)^2 = \frac{k^2m}{s_1}$$

If the particle were stopped by striking a body when it reached the point s_1, its kinetic energy would be changed into some other form, principally heat It has been found by experiment that a body weighing one kilogram falling 425 meters * in the vicinity of the earth's surface, under the influence of the earth's attraction, will generate enough heat to raise the temperature of one kilogram of water one degree Centigrade This quantity of heat is called the *calorie* The amount of heat generated is proportional to the product of the square of the velocity and the mass of the moving particle Then, letting Q represent the number of calories, it follows that

$$(44)\qquad Q = Cmv^2$$

Let m be expressed in kilograms and v in meters per second, in order to determine the constant C Take Q and m each equal to

* Joule found 423 5, Rowland 427 8 For results of experiments and references see Preston's *Theory of Heat*, p 594

unity, then the velocity is that acquired by the body falling through 425 meters It was shown in Art 30 that, if the body falls from rest,

$$\begin{cases} s = \frac{1}{2}gt^2, \\ v = gt \end{cases}$$

Eliminating t, it follows that

$$v^2 = 2gs$$

In the units employed $g = 9\ 8094$, and since $s = 425$, $v^2 = 8338$, and by (44),

$$C = \tfrac{1}{8338}$$

Then the general formula (44) becomes

$$Q = \frac{mv^2}{8338}, \tag{45}$$

where Q is expressed in calories if the kilogram, meter, and second are taken as the units of mass, distance, and time

42 The Temperature of Meteors The increase of temperature of a body, when the proper units are employed, is equal to the number of calories of heat acquired divided by the product of the mass and the specific heat of the substance Suppose a meteor whose specific heat is unity (in fact it would probably be much less) and whose mass is one kilogram should strike the earth with any given velocity, it is required to compute the increase of temperature to which it would be subject if it took up all the heat generated The specific heat has been taken so that the increase of temperature is numerically equal to the number of calories generated Meteors usually strike the earth with a velocity of about 25 miles, or 40233 meters, per second Substituting for v 40233 and for m unity in (45), it is found that $T = Q = 194134$, the number of calories generated, or the number of degrees through which the temperature of the meteor would be raised Practically a large part of the heat would be imparted to the atmosphere which would destroy most of the velocity with which the meteor would fall, but the quantity of heat is so enormous that it could not be expected that any but the largest meteors would last long enough to reach the earth

A meteor falling into the sun from an infinite distance would strike its surface, as has been seen in Art 36, with a velocity of about 380 miles per second The heat generated would be therefore $(\frac{380}{25})^2$, or 231, times as great as that produced in striking the earth Thus it follows that a kilogram would generate, in falling into the sun, 44,844,954 calories

43 The Meteoric Theory of the Sun's Heat When it is remembered what an enormous number of meteors (estimated by the late Dr H A Newton* as being 8,000,000 daily) strike the earth, it is easily conceivable that enough strike the sun to maintain its temperature Indeed, this has been advanced as a theory to account for the replenishment of the vast amount of heat which the sun radiates There can be no question of its qualitative correctness, and it only remains to examine it quantitatively

Let it be assumed that the sun radiates heat equally in every direction, and that meteors fall upon it in equal numbers from every direction Therefore, the amount of heat radiated by any portion of the surface will equal the amount generated by the impact of meteors upon that portion The amount of heat received by the earth will be to the whole amount radiated from the sun as the surface which the earth subtends as seen from the sun is to the surface of the whole sphere whose radius is the distance from the earth to the sun The portion of the sun's surface which is within the cone whose base is the earth and vertex the center of the sun, is to the whole surface of the sun as the surface subtended by the earth is to the surface of the whole sphere whose radius is the distance to the sun Therefore, the earth receives as much heat as is radiated by, and consequently generated upon, the surface cut out by this cone But the earth would intercept precisely as many meteors as would fall upon this small area, and would, therefore, receive heat from the impact of a certain number of meteors upon itself, and as much heat from the sun as would be generated by the impact of an equal number upon the sun

The heat derived by the earth from the two sources would be as the squares of the velocities with which the meteors strike the earth and sun It was seen in Art 42 that this number is $\frac{1}{231}$ Therefore, if this meteoric hypothesis of the maintenance of the sun's heat is correct the earth should receive $\frac{1}{231}$ as much heat from the impact of meteors as from the sun This is certainly millions of times more heat than the earth receives from meteors, and consequently the theory is not tenable

44 Helmholtz's Contraction Theory The amount of work done upon a particle is proportional to the product of the resistance overcome by the distance moved There is nothing whatever said about how long the motion shall take, and if the work is converted into heat the total amount will be the same whether the particle falls

* *Mem Nat Acad of Sci*, vol I

the entire distance at once, or covers the same distance by a succession of any number of shorter falls When a body contracts it is equivalent to a succession of very short movements of all its particles in straight lines toward the center, and it is evident that, knowing the law of density, the amount of heat which will be generated can be computed

In 1854 Helmholtz applied this idea to the computation of the heat of the sun in an attempt to explain its source of supply He made the supposition that the sun contracts in such a manner that it always remains homogeneous While this assumption is certainly incorrect, nevertheless the results obtained are of great value and give a good idea of what doubtless actually takes place under contraction The mathematical part of the theory is given in the *Philosophical Magazine* for 1856, p 516

Consider a homogeneous gaseous sphere whose radius is R_0 and density σ Let M_0 represent its mass Let dM represent an element of mass taken anywhere in the interior or at the surface of the sphere Let R be the distance of dM from the center of the sphere, and let M represent the mass of the sphere whose radius is R The element of mass in polar coordinates is (Art 21)

$$dM = \sigma R^2 \cos\phi \, d\phi \, d\theta \, dR \tag{46}$$

The element is subject to the attraction of the whole sphere within it As will be shown in Chapter IV the attraction of the spherical shell outside of it balances in opposite directions so that it need not be considered in discussing the forces acting upon dM Every element in the infinitesimal shell whose radius is R is attracted toward the center by a force equal to that acting on dM, therefore the whole shell may be treated at once Let dM_s represent the mass of the elementary shell whose radius is R It is found by integrating (46) with respect to θ and ϕ Thus

$$dM_s = \sigma R \, dR \int_0^{2\pi} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi \, d\phi \right\} d\theta = 4\pi\sigma R^2 dR \tag{47}$$

The force to which dM_s is subject is $-\dfrac{k^2 M dM_s}{R^2}$ The element of work done in moving dM_s through the element of distance dR is

$$dW_s = -dM_s \frac{k^2 M}{R^2} dR$$

The work done in moving the shell from the distance CR to R is the integral of this expression between the limits CR and R, or

$$W_s = -dM_s k^2 M \int_{CR}^{R} \frac{dR}{R^2} = \frac{dM_s k^2 M}{R}\left(\frac{C-1}{C}\right)$$

But $M = \frac{4}{3}\pi\sigma R^3$, hence, substituting the value of dM_s from (47) and representing the work done on the elementary shell by $W_s = dW$, it follows that

$$dW = \tfrac{16}{3}\pi^2\sigma^2 k^2 \left(\frac{C-1}{C}\right) R^4 dR$$

The integral of this expression from 0 to R_0 gives the total amount of work done in the contraction of the homogeneous sphere from radius CR_0 to R_0 That is,

$$W = \tfrac{16}{3}\pi^2\sigma^2 k^2 \left(\frac{C-1}{C}\right) \int_0^{R_0} R^4 dR = \tfrac{16}{15}\pi^2\sigma^2 k^2 \left(\frac{C-1}{C}\right) R_0^5,$$

which may be written

$$(48) \qquad W = \tfrac{3}{5}k^2 \left(\frac{C-1}{C}\right)\frac{M_0^2}{R_0}$$

If C equals infinity

$$(49) \qquad W = \tfrac{3}{5}k^2 \frac{M_0^2}{R_0}$$

If the second be taken as the unit of time, the kilogram as the unit of mass, and the meter as the unit of distance, and if k^2 be computed from the value of g for the earth, then after dividing by $\frac{8338}{2}$, W will be numerically equal to the amount of heat in calories that will be generated if the work is all transformed into this kind of energy The temperature to which the body will be raised is

$$(50) \qquad T = \frac{H}{M_0\eta} = \frac{2W}{8338 M_0\eta},$$

where η is the specific heat of the substance Or, substituting (49) in (50),

$$(51) \qquad T = \frac{3k^2}{5\eta}\ \frac{C-1}{C}\ \frac{M_0}{R_0}\ \frac{2}{8338}$$

By definition k^2 is the attraction of the unit of mass at unit distance, therefore, if m is the mass of the earth and r its radius, it follows that

$$g = \frac{k^2 m}{r^2}$$

Solving for k^2 and substituting in (51), this equation becomes

$$(52) \qquad T = \frac{3(C-1)}{5\eta C}\ \frac{r^2}{R_0}\ \frac{M_0}{m}\ \frac{2g}{8338}$$

If the body contracted from infinity ($C = \infty$), the amount of heat

generated would be sufficient to raise its temperature T degrees Centigrade, where T is given by the equation

$$(53)\qquad T=\frac{3}{5}\,\frac{1}{\eta}\,\frac{r^2}{R_0}\,\frac{M_0}{m}\,\frac{2g}{8338}$$

Suppose the specific heat is taken as unity, which is that of water* The following data have been taken from the *Annuaire du Bureau des Longitudes* for 1900

$$\begin{cases} r=6{,}371{,}000 \\ R=692{,}428{,}000 \\ M=324{,}439 \\ m=1 \\ g=9\ 8094 \end{cases}$$

Substituting in (53) and reducing, it is found that

$$T=26{,}850{,}000^\circ \text{ Centigrade}$$

Therefore, *the sun contracting from infinity in such a way as to always remain homogeneous would generate enough heat to raise the temperature of an equal mass of water more than twenty-six millions of degrees Centigrade*

If it is supposed that the sun has contracted only from Neptune's orbit equation (52) may be used, which will give a value of T about $\frac{1}{6000}$ less In any case it is not intended to imply that it did ever contract from such great dimensions in the particular manner assumed the results given are nevertheless significant and throw much light on the possible mode of evolution of the solar system from a vastly extended nebula

The experiments of Pouillet† have shown that, under the assumption that the sun radiates heat equally in every direction, the amount of heat emitted yearly would raise the temperature of a mass of water equal to that of the sun 1 25 degrees Centigrade In order to find how great a shrinkage in the present radius would be required to generate enough heat to maintain the present radiation 10,000 years substitute 12,500 for T in (52) and solve for C Carrying out the computation, it is found that

$$C=1\ 000465$$

Therefore, *the sun would generate enough heat in shrinking less than one two thousandth of its present diameter to maintain its present radiation* 10,000 *years*

* No other ordinary terrestrial substance has a specific heat so great as unit except hydrogen gas, whose specific heat is 3 409 But the lighter gases of the sola atmosphere may also have high values

† *Comptes Rendus*, 1838

The sun's mean apparent diameter is 1924″, so a contraction of its diameter of 000465 would make an apparent change of only 0″ 9, a quantity far too small to be observed on such an object by the methods now in use Reducing the shrinkage to other units, it is found that a contraction of the sun's radius of 32 2 meters annually would account for all the heat that is being radiated at present

VI PROBLEMS

1 According to Young's *General Astronomy*, Art 338, a square meter exposed perpendicularly to the sun's rays at the earth's distance would receive 30 calories per minute* The average amount received per square meter on the earth's surface is to this quantity as the area of a circle is to the surface of a sphere of the same radius, or 1 to 4 The earth's surface receives, therefore, on the average $7\frac{1}{2}$ calories per square meter per minute How many kilograms of meteoric matter would have to strike the earth with a velocity of 25 miles (40233 meters) per sec to generate $\frac{1}{231}$ this amount of heat?

Ans 000,000,1673 kilograms

2 How many pounds would have to fall per day on every square mile on the average? Tons on the whole earth?

Ans $\begin{cases} \text{1375 pounds} \\ \text{135,400,000 tons} \end{cases}$

3 Find the amount of work done in the contraction of any fraction of the radius of a sphere when the law of density is $\sigma = \frac{l}{R^2}$

Ans $W = 16\pi^2 k^2 l^2 \left(\frac{C-1}{C}\right) R = k^2 \left(\frac{C-1}{C}\right) \frac{M_0^2}{R_0}$, or $\frac{5}{3}$ of the work done when the sphere is homogeneous

4 Find the amount of work done in the contraction of a spherical shell from radius CR_0 to R_0 whose density is $\sigma = \frac{l}{R^n}$, around a homogeneous solid nucleus of radius A and density $\frac{l}{A^n}$ For what values of n does the general formula fail?

Ans $$W = \frac{16\pi^2 l^2 k^2}{3(n-2)(2n-5)A^{n-3}} \left(\frac{C-1}{C}\right) \left[\frac{2(n-1)}{A^{n-2}} + \frac{3(n-2)A^{n-3}}{(n-3)R_0^{2n-5}} - \frac{2n-5}{(n-3)R_0^{n-2}}\right]$$

5 Find how much the heat generated by the contraction of the earth from the density of meteorites, 3 5, to the present density of 5 6 would raise the temperature of the whole earth, assuming that the specific heat is 0 2

Ans $T = 6520\ 5$ degrees Centigrade

* Langley's results

HISTORICAL SKETCH AND BIBLIOGRAPHY

The laws of falling bodies under constant acceleration were investigated by Galileo and Stevinus, and for many cases of variable acceleration by Newton Such problems are comparatively simple when treated by the analytical processes which have come into use since the time of Newton Parabolic motion was discussed by Galileo and Newton

The kinetic theory of gases seems to have been first suggested by J Bernoulli about the middle of the 18th century, but it was first developed mathematically by Clausius Maxwell, Boltzmann, and O E Meyer have made important contributions to the subject, and more recently Burbury Some of the principal books on the subject are Risteen's *Molecules and the Molecular Theory* (descriptive work), L Boltzmann's *Gastheorie*, H W Watson's *Kinetic Theory of Gases*, O E Meyer's *Die Kinetische Theorie der Gase*, S H Burbury's *Kinetic Theory of Gases*

The meteoric theory of the sun's heat was first suggested by R Mayer The contraction theory was first announced in a public lecture by Helmholtz at Konigsberg Feb 7, 1854 and was published later in *Phil Mag* 1856 An important paper by J Homer Lane appeared in the *Am Jour of Sci* July, 1870 The amount of heat generated depends upon the law of density of the gaseous sphere Investigations covering this point are 16 papers by Ritter in *Wiedemann's Annalen*, vol v 1878 to vol xx 1883, by G W Hill, *Annals of Math* vol iv 1888, and by G H Darwin, *Phil Mag* 1888 The original papers must be read for an exposition of the subject of the heat of the sun

CHAPTER III

CENTRAL FORCES

45 Central Force This chapter will be devoted to the discussion of the motion of a material particle when subject to an attractive or repelling force whose line of action always passes through a fixed point This fixed point will be called the *center of force* It is not implied that the force emanates from the center or that there is but one force, but simply that the resultant of all the forces acting on the particle always passes through this point The force may be directed toward the point or from it, or part of the time toward and part of the time from it It may be zero at any time, but if the particle passes through a point where the force to which it is subject becomes infinite, a special investigation which cannot be taken up here, is required to follow it farther Since attractive forces are of most frequent occurrence in astronomical and physical problems the formulas developed will be for this case, a change of sign of the coefficient of intensity of force for unit distance will make the formulas valid for the case of repulsion

The origin of coordinates will be taken at the center of force, and the line from the origin to the moving particle will be called the *radius vector* The path described by the particle will be called the *orbit* The orbits of this chapter are plane curves The planes are defined by the position of the center of force and the line of initial projection The xy-plane will be taken uniformly as the plane of the orbit

46 The Law of Areas The first problem will be to derive the general properties of motion which hold for all central forces The first of these, which is of great importance, is the *law of areas*, and constitutes the first Proposition of Newton's *Principia* It is, *if a particle is subject to any central force, the areas which are swept over*

by the radius vector are proportional to the intervals of time in which they are described The following is Newton's demonstration of it

Let O be the center of force, and let the particle be projected from A in the direction of B with the velocity AB Then, by the first law of motion, it would pass to C' in the first two units of time if there were no external forces acting upon it But suppose that when it arrives at B an instantaneous force acts upon it in the direction of the origin with such intensity that it would move to b in a unit of time if

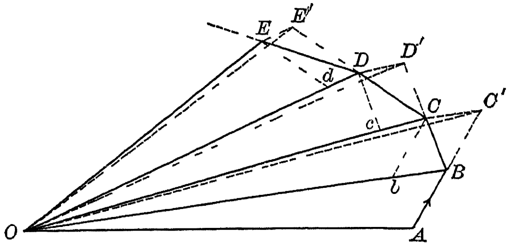

Fig 7

it had no previous motion Then, by the second law of motion, it will move along the diagonal of the parallelogram $BbCC'$ to C If no other force were applied it would move with uniform velocity to D' in the next unit of time But suppose that when it arrives at C another instantaneous force acts upon it in the direction of the origin with such intensity that it would move to c in a unit of time if it had no previous motion Then, as before, it will move along the diagonal of the parallelogram and arrive at D at the end of the unit of time This process may be repeated indefinitely

The following equalities among the areas of the triangles involved hold, since they have sequentially equal bases and altitudes,

$$OAB = OBC' = OBC = OCD' = OCD = \text{etc}$$

Therefore, it follows that $OAB = OBC = OCD = ODE$, etc That is, the areas of the triangles swept over in the succeeding units of time are equal, and, therefore, the sums of the areas of the triangles described in any intervals of time are proportional to the intervals

The reasoning is true without any changes however small the unit of time is taken, and however frequent the instantaneous impulses are supposed to act Let the path from A to some other point E be considered Suppose the unit of time is taken shorter and shorter the impulses will become closer and closer together, and the broken

line will become nearer and nearer a smooth curve. Suppose now the unit of time approaches zero as a limit, the succession of impulses will approach a continuous force as a limit, and the broken line will approach a smooth curve as a limit. The areas swept over by the radius vector in any finite intervals of time are proportional to these intervals during the whole limiting process. Therefore, the proportionality of areas holds at the limit, and the theorem is true for a continuous central force.

It will be observed that it is not necessary that the central force shall vary continuously. It may be attractive and instantaneously change to repulsion, or become zero, and the law will still hold; but it is necessary to exclude the case where it becomes infinite unless a special investigation is made.

The linear velocity varies inversely as the perpendicular from the origin upon the tangent to the curve at the point of the moving particle; for, the area described in a unit of time is equal to the product of the velocity and the perpendicular upon the tangent. Since the area described in a unit of time is always the same, it follows that the velocity varies inversely as the perpendicular.

47. Analytical Demonstration. It is of interest to note that, although the language of Geometry was employed in the demonstration above, yet the essential elements of the methods of the Differential and Integral Calculus were used. Thus, in passing to the limit, the process was essentially that of expressing the problem in differential equations; and, in insisting upon comparing intervals of finite size when the units of measurement were indefinitely decreased, the process of integration was really employed. It will be found that in the treatment of all problems involving variable forces and motions the methods are in essence those of the Calculus, even though the demonstrations be couched in geometrical language. It is perhaps easier to follow the reasoning in geometrical form when one encounters it for the first time; but the processes are all special and involve fundamental difficulties which are often troublesome. On the other hand, the development of the Calculus is of the precise form to adapt it to the treatment of these problems, and after its principles have been once mastered, the application of it is characterized by comparative simplicity and great generality. A few problems will be treated by both methods to show their essential sameness, and to illustrate the advantages of analysis.

Let f represent the acceleration to which the particle is subject. In the problems which will be treated the acceleration will depend

upon the position of the particle, and not upon the time or velocity with which it moves By hypothesis the line of force always passes through a fixed point, which will be taken as the origin of coordinates

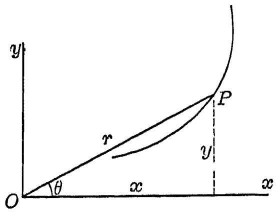

Fig 8

Let O be the center of force, and P any position of the particle with the rectangular coordinates x and y, and the polar coordinates r and θ Then the components of acceleration along the x and y-axes are respectively $\mp f \cos\theta$, and $\mp f \sin\theta$, and the differential equations of motion are

$$(1)\qquad \begin{cases} \dfrac{d^2x}{dt^2} = \mp f \cos\theta = \mp f \dfrac{x}{r}, \\[2ex] \dfrac{d^2y}{dt^2} = \mp f \sin\theta = \mp f \dfrac{y}{r} \end{cases}$$

The negative sign must be taken before the right members of these equations if the force is attractive, and the positive if it is repulsive

Multiply the first equation of (1) by $-y$ and the second one by x and add The result is

$$x\frac{d^2y}{dt} - y\frac{d^2x}{dt} = 0$$

Integrating this expression by parts, it follows that

$$(2)\qquad x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

where h is the constant of integration

The integrals of differential equations generally lead to important theorems even though the whole problem has not been solved, and in the following they will be discussed as they are obtained

Referring to Art 16, it is seen that (2) may be written

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r\frac{d\theta}{dt} = 2\frac{dA}{dt} = h,$$

where A is the area swept over by the radius vector The integral of this equation is

$$A = \frac{h}{2}t + c,$$

shows that the area is directly proportional to the time, which is
ieorem to be proved

3 **Converse of the Theorem of Areas** By hypothesis

$$A = c_1 t + c_2$$

g the derivative with respect to t, it is found that

$$\frac{dA}{dt} = c_1$$

becomes in polar coordinates

$$r^2 \frac{d\theta}{dt} = 2c_1,$$

ı rectangular coordinates

$$x \frac{dy}{dt} - y \frac{dx}{dt} = 2c_1$$

erivative of this expression with respect to t is

$$x \frac{d^2y}{dt^2} - y \frac{d^2x}{dt^2} = 0,$$

$$\frac{d^2x}{dt^2} \quad \frac{d^2y}{dt^2} = x \quad y$$

is, the components of acceleration are proportional to the co-
ıtes, therefore, if the law of areas is true with respect to a point
sultant of the acceleration passes through that point

, if $r^2 \frac{d\theta}{dt} = 2c_1$, then $\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0$, and by (19), Art 14, the
ration perpendicular to the radius vector is zero, that is, the
ration is all in a line passing through the origin

) **The Laws of Angular and Linear Velocity** From the
ssion for the law of areas in polar coordinates, it follows that

$$\frac{d\theta}{dt} = \frac{h}{r^2},$$

ore, *the angular velocity is inversely proportional to the square of
dius vector*

ıe linear velocity is

$$\frac{ds}{dt} = \frac{ds}{d\theta}\frac{d\theta}{dt} = \frac{ds}{d\theta}\frac{h}{r^2}$$

represent the perpendicular from the tangent upon the origin,
it is known from Differential Calculus that*

$$\frac{ds}{d\theta} = \frac{r^2}{p}$$

* Williamson's *Differential Calculus*, p 224

Hence the expression for the linear velocity becomes

$$\text{(4)} \qquad \frac{ds}{dt} = \frac{h}{p},$$

therefore, *the linear velocity is inversely proportional to the perpendicular from the origin upon the tangent*

Simultaneous Differential Equations

50 The Order of a System of Simultaneous Differential Equations* One integral, equation (2), of the differential equations (1) has been found which the motion of the particle must fulfill The question is how many more integrals must be found in order to have the complete solution of the problem

The number of integrals which must be found to completely solve a system of differential equations will be called the *order* of the system Thus, the equation

$$\text{(5)} \qquad \frac{d^n x}{dt^n} = c$$

is of the nth order, because it must be integrated n times to be reduced to an integral form Similarly, the general equation

$$\text{(6)} \qquad f_n \frac{d^n x}{dt^n} + f_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + f_1 \frac{dx}{dt} + f_0 = 0,$$

where $f_n, \ldots, f_0$ are functions of x and t, must be integrated n times in order to express x as a function of t, and is of the nth order

An equation of the nth order may be reduced to an equivalent system of n simultaneous equations each of the first order Thus, to reduce (6) to a simultaneous system, let

$$x_1 = \frac{dx}{dt}, \quad x_2 = \frac{dx_1}{dt}, \quad \ldots, \quad x_{n-1} = \frac{dx_{n-2}}{dt},$$

whence

$$\text{(7)} \qquad \begin{cases} \dfrac{dx}{dt} = x_1, \\ \dfrac{dx_1}{dt} = x_2, \\ \dfrac{dx_2}{dt} = x_3, \\ \quad \vdots \\ \dfrac{dx_{n-1}}{dt} = -\dfrac{f_{n-1}}{f_n} x_{n-1} - \cdots - \dfrac{f_1}{f_n} x_1 - \dfrac{f_0}{f_n} \end{cases}$$

* See Jordan's *Cours d'Analyse*, vol III chap I

Therefore, these n simultaneous equations, each of the first order, constitute a system of the nth order An equation, or a system of equations, reduced to the form (7) is said to be reduced to the *normal form*, and the system is called a *normal system*

Two simultaneous equations of order m and n may be reduced to a normal system of order $m+n$ Thus, consider the equations

$$(8) \qquad \begin{cases} f_m \dfrac{d^m x}{dt^m} + \quad + f_1 \dfrac{dx}{dt} + f_0 = 0, \\ \phi_n \dfrac{d^n y}{dt^n} + \quad + \phi_1 \dfrac{dy}{dt} + \phi_0 = 0, \end{cases}$$

where f_i and ϕ_i are functions of x, y, and t By a substitution similar to that used above, it follows that they are equivalent to the normal system

$$(9) \qquad \begin{cases} \dfrac{dx}{dt} = x_1, \\ \\ \dfrac{dx_{m-1}}{dt} = -\dfrac{f_{m-1}}{f_m} x_{m-1} - \quad - \dfrac{f_1}{f_m} x_1 - \dfrac{f_0}{f_m}, \\ \dfrac{dy}{dt} = y_1, \\ \\ \dfrac{dy_{n-1}}{dt} = -\dfrac{\phi_{n-1}}{\phi_n} y_{n-1} - \quad - \dfrac{\phi_1}{\phi_n} y_1 - \dfrac{\phi_0}{\phi_n}, \end{cases}$$

which is of the order $m+n$ Evidently a similar reduction is possible when each equation contains derivatives with respect to both of the variables, either separately or as products

Conversely, *a normal system of order n may be transformed into a single equation of order n with one dependent variable* To fix the ideas, consider a system of the second order, as

$$(10) \qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = \phi(x, y, t) \end{cases}$$

In addition to these two equations form the derivative of one of them with respect to t, as

$$(11) \qquad \frac{d^2x}{dt^2} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$$

From equations (10) and (11) y and $\frac{dy}{dt}$ may be eliminated, giving an equation of the form

$$f_2 \frac{d^2x}{dt^2} + f_1 \frac{dx}{dt} + f_0 = 0$$

If the normal system were of the third order in the dependent variables x, y, and z, the first and second derivatives of the first equation would be taken, and the first derivative of the second and third equations These four new equations with the original three would make seven from which y, z, $\frac{dy}{dt}$, $\frac{dz}{dt}$, $\frac{d^2y}{dt^2}$, and $\frac{d^2z}{dt^2}$ could be eliminated, giving an equation of the third order in x alone This process may be extended to a system of any order

The differential equations (1) may be reduced by the substitution $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$ to the normal system

$$(12) \qquad \begin{cases} \frac{dx}{dt} = x, & \frac{dx}{dt} = \mp f \frac{x}{r}, \\ \frac{dy}{dt} = y', & \frac{dy'}{dt} = \mp f \frac{y}{r}, \end{cases}$$

which is of the fourth order Therefore four integrals must be found in order to have the complete solution of the problem The components of velocity, x' and y', play rôles similar to the coordinates in (12), and, for brevity, they will be spoken of frequently in the future as being coordinates

51 Reduction of Order When an integral of a system of differential equations has been found two methods may be followed in completing the solution The remaining integrals may be found from the original differential equations as though none was already known, or, by means of the known integral, the order of the system of differential equations may be reduced by one That the order of the system may be reduced by means of the known integrals will be shown in the general case Consider the system

$$(13) \qquad \begin{cases} \frac{dx_1}{dt} = f_1(x_1, \quad, x_n, t), \\ \frac{dx_2}{dt} = f_2(x_1, \quad, x_n, t), \\ \\ \frac{dx_n}{dt} = f_n(x_1, \quad, x_n, t) \end{cases}$$

Suppose an integral

$$F(x_1, x_2, \quad, x_n, t) = \text{constant}$$

has been found

Solve this equation for one of the variables, as

$$x_1 = \psi(x_2, x_3, \quad, x_n, t)$$

Substitute this expression for x_1 in the last $n-1$ equations of (13), they become

$$(14) \qquad \begin{cases} \dfrac{dx_2}{dt} = \phi_2(x_2, \quad, x_n, t), \\ \dfrac{dx_3}{dt} = \phi_3(x_2, \quad, x_n, t), \\ \\ \dfrac{dx_n}{dt} = \phi_n(x_2, \quad, x_n, t) \end{cases}$$

This is a simultaneous system of order $n-1$, and is independent of the variable x_1

It is apparent from these theorems and remarks that the order of a simultaneous system of differential equations is equal to the sum of the orders of the individual equations, that the equations may be written in several ways, e g as one equation of the nth order, or n equations of the first order, and that the integrals may all be derived from the original system, or that the order may be reduced after each integral is found In mechanical and physical problems the intuitions are important in suggesting methods of treatment, so it is generally advantageous to use such variables that their geometrical meanings shall be easily perceived For this reason, it is generally simpler not to reduce the order of the problem after each integral is found

VII PROBLEMS

1 Prove the converse of the law of areas by the geometrical method, and show that the steps agree with the analysis of Art 48

2 Prove the law of angular velocity by the geometrical method

3 Why cannot equations (1) be integrated separately?

4 Derive the law of areas directly from equation (2) without passing to polar coördinates

5 Show in detail that a normal system of order four may be reduced to a single equation of order four, and the converse

6 Reduce the system of equations (12) to one of the third order by means of the integral of areas

52 The Vis Viva Integral Suppose the acceleration is toward the origin, then the negative sign must be taken before the right members of equations (1) Multiply the first of (1) by $2\,\frac{dx}{dt}$, the second by $2\,\frac{dy}{dt}$, and add The result is

$$2\,\frac{d^2x}{dt^2}\,\frac{dx}{dt}+2\,\frac{d^2y}{dt^2}\,\frac{dy}{dt}=-\frac{2f}{r}\left(x\,\frac{dx}{dt}+y\,\frac{dy}{dt}\right)$$

From $r^2=x^2+y^2$ it follows that

$$x\,\frac{dx}{dt}+y\,\frac{dy}{dt}=r\,\frac{dr}{dt},$$

therefore

$$2\,\frac{d^2x}{dt^2}\,\frac{dx}{dt}+2\,\frac{d^2y}{dt^2}\,\frac{dy}{dt}=-2f\frac{dr}{dt}$$

Suppose f depends upon r alone, as it does in most astronomical and physical problems Then $f=\phi(r)$, and

$$2\,\frac{d^2x}{dt^2}\,\frac{dx}{dt}+2\,\frac{d^2y}{dt}\,\frac{dy}{dt}=-2\phi(r)\,\frac{dr}{dt}$$

The integral of this equation is

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)=v^2=-2\int\phi(r)\,dr+c \tag{15}$$

When the form of the function $\phi(r)$ is given the integral on the right can be found Suppose it is $\Phi(r)$, then

$$v^2=-2\Phi(r)+c \tag{16}$$

If $\Phi(r)$ is a single-valued function of r, as it is in physical problems, it follows from (16) that, if the central force is a function of the distance alone, the velocity is the same at all points equally distant from the origin Its magnitude depends upon the initial distance and velocity, and not upon the path described Since the force of gravitation varies inversely as the square of the distance of the attracting bodies apart, it follows that when a body is approaching the sun it moves with the same velocity as when receding at the same distance

Examples where f is a Function of the Coordinates

53 Force Varying Directly as the Distance In order to find integrals of equations (1) other than that of areas, the value of f in terms of the coordinates must be known There is one case in which the integration becomes particularly simple It is when the intensity of the force varies directly as the distance Let k^2 represent

the acceleration at unit distance Then $f = k^2 r$, and equations (1) become

$$(17)\qquad \begin{cases} \dfrac{d^2x}{dt^2} = -k^2 x, \\ \dfrac{d^2y}{dt^2} = -k^2 y \end{cases}$$

An important property of these equations is that each is independent of the other, as the first one contains the dependent variable x alone and the second one y alone It is observed, moreover, that they are linear and the solution may be found by the method given in Art. 32 The results are, for the first equation,

$$(18)\qquad \begin{cases} x = c_1 e^{\sqrt{-1}kt} + c_2 e^{-\sqrt{-1}kt}, \\ \dfrac{dx}{dt} = \sqrt{-1}\,k\left(c_1 e^{\sqrt{-1}kt} - c_2 e^{-\sqrt{-1}kt}\right) \end{cases}$$

Suppose $x = x_0$, $\dfrac{dx}{dt} = x_0'$, when $t = 0$, then

$$\begin{cases} x_0 = c_1 + c_2, \\ x_0' = \sqrt{-1}\,k\,(c_1 - c_2), \end{cases}$$

whence

$$\begin{cases} 2c_1 = x_0 + \dfrac{x_0'}{\sqrt{-1}\,k}, \\ 2c_2 = x_0 - \dfrac{x_0'}{\sqrt{-1}\,k} \end{cases}$$

Therefore, equations (18) become

$$\begin{cases} x = \dfrac{x_0}{2}\left(e^{\sqrt{-1}kt} + e^{-\sqrt{-1}kt}\right) + \dfrac{x_0'}{2k\sqrt{-1}}\left(e^{\sqrt{-1}kt} - e^{-\sqrt{-1}kt}\right), \\ \dfrac{dx}{dt} = \dfrac{\sqrt{-1}\,kx_0}{2}\left(e^{\sqrt{-1}kt} - e^{-\sqrt{-1}kt}\right) + \dfrac{x_0'}{2}\left(e^{\sqrt{-1}kt} + e^{-\sqrt{-1}kt}\right), \end{cases}$$

or, in the trigonometrical form,

$$(19)\qquad \begin{cases} x = x_0 \cos kt + \dfrac{x_0'}{k} \sin kt, \\ \dfrac{dx}{dt} = -x_0 k \sin kt + x_0' \cos kt, \quad \text{and similarly,} \\ y = y_0 \cos kt + \dfrac{y_0'}{k} \sin kt, \\ \dfrac{dy}{dt} = -y_0 k \sin kt + y_0' \cos kt \end{cases}$$

The equation of the orbit is obtained by eliminating t from the

first and third equations Multiplying by the appropriate factors and adding, it is found that

$$\begin{cases}(x_0 y_0' - y_0 x_0') \sin kt = k\,(x_0 y - y_0 x), \\ (x_0 y_0' - y_0 x_0') \cos kt = y_0' x - x_0' y\end{cases}$$

Squaring and adding,

$$(20)\qquad (k^2 y_0^2 + y_0'^2)\,x^2 + (k^2 x_0^2 + x_0'^2)\,y^2 - 2\,(k^2 x_0 y_0 + x_0' y_0')\,xy = (x_0 y_0' - y_0 x_0')^2$$

This is the equation of an ellipse with the origin at the center unless $x_0 y_0' - y_0 x_0' = 0$, when it degenerates to two straight lines which must be coincident, for, then

$$\frac{x_0}{x_0'} = \frac{y_0}{y_0'} = \text{constant} = c\,,$$

from which

$$x_0 = c x_0', \quad y_0 = c y_0'$$

In this case equation (20) becomes

$$(21)\qquad (k^2 c^2 + 1)\,(y_0' x - x_0' y)^2 = 0,$$

and the motion is rectilinear and oscillatory In every case both the coordinates and the components of velocity change with a period of $\frac{2\pi}{k}$, whatever the initial conditions may be

54 Differential Equation of the Orbit The curve described by the moving particle will be in all cases of much interest A general method of finding it is to integrate the differential equations and then eliminate the time This is often a complicated process, and the question arises whether the time may not be eliminated before the integration is performed, so that the integration will lead directly to the orbit It will be shown that this is the case

Consider the equations

$$(22)\qquad \begin{cases}\dfrac{d^2x}{dt^2} = -f\dfrac{x}{r}, \\[2ex] \dfrac{d^2y}{dt^2} = -f\dfrac{y}{r}\end{cases}$$

Since f depends upon the coordinates alone the time enters only in the derivatives But a second differential quotient cannot be separated as though it were an ordinary fraction, therefore, the order of the derivatives must be reduced before the direct elimination of t can be made, or an indirect process must be employed The latter method will be adopted

$\frac{1}{u}$, therefore

$$x = \frac{\cos\theta}{u}$$

ive with respect to t is

$$= -\frac{\sin\theta}{u}\frac{d\theta}{dt} - \frac{\cos\theta}{u^2}\frac{du}{dt} = -\left(u\sin\theta + \cos\theta\frac{du}{d\theta}\right)\frac{1}{u^2}\frac{d\theta}{dt}$$

ntegral of areas,

$$\frac{1}{u^2}\frac{d\theta}{dt} = h$$

g this value of h and differentiating again with respect to t, 3) becomes

$$\frac{d^2x}{dt^2} = -h\left\{u\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{du}{dt} - \sin\theta\frac{du}{d\theta}\frac{d\theta}{dt} + \cos\theta\frac{d\left(\frac{du}{d\theta}\right)}{dt}\right\}$$

$$\frac{d\left(\frac{du}{d\theta}\right)}{dt} = \frac{d\left(\frac{du}{d\theta}\right)}{d\theta}\frac{d\theta}{dt} = \frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$$

s substitution and eliminating $\frac{d\theta}{dt}$ by means of the equation 4) becomes

$$\frac{d^2x}{dt^2} = -h^2u^2\left(u\cos\theta + \cos\theta\frac{d^2u}{d\theta^2}\right)$$

e right member of this equation equal to the right member of (22), it is found that

$$f = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

differential equation of the orbit in polar coordinates It econd order, but one integral has been used in determining re the problem of finding the path of the body is of the The complete problem was of the fourth order, the fourth presses the relation between the coordinates and the time, he position of the particle in the orbit

sely, equation (25) may be used to find the law of central will cause a particle to describe a given curve It is only o write the equation of the curve in polar coordinates and to ie right member This is generally a simpler process than one of finding the orbit when the law of force is given

55 Derivation of Newton's Law In the early part of the seventeenth century Kepler announced three laws of planetary motion He derived them from a most laborious discussion of a long series of observations of the planets, especially of Mars They are the following

LAW I *The radius vector of each planet with respect to the sun as the origin, sweeps over equal areas in equal times*

LAW II *The orbit of each planet is an ellipse with the sun at one of its foci*

LAW III *The squares of the periods of the planets are to each other as the cubes of the major semi-axes of their respective orbits*

It was on these laws that Newton based his demonstration that the planets move subject to forces directed toward the sun, and varying inversely as the squares of their distances from the sun The Newtonian law will be derived here by employing the analytical method instead of the geometrical methods of the *Principia**

From the converse of the theorem of areas and Kepler's first law, it follows that the planets move subject to central forces directed toward the sun The curves described are given by the second law, and equation (25) may, therefore, be used to find the expression for the acceleration in terms of the coordinates Let a represent the major semi-axis of the ellipse, and e its eccentricity, then its equation in polar coordinates is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

Therefore
$$u + \frac{d^2u}{d\theta^2} = \frac{1}{a(1-e^2)}$$

Substituting in (25), it is found that the expression for the acceleration is

$$f = \frac{h}{a^2(1-e)} \frac{1}{r^2} = \frac{k^2}{r}$$

Therefore, the acceleration to which any planet is subject varies inversely as the square of its distance from the sun

The third law shows, as will be proved later†, that the planets are all subject to the same acceleration when reduced to unit distance

From the consideration of these laws, the gravity at the earth's

* Book I, Proposition XI

† Art 89

surface, and the motion of the moon around the earth, Newton was led to the enunciation of the Law of Universal Gravitation, which is, *every particle of matter in the universe attracts every other particle with a force which acts in a line joining them, and whose intensity varies as the product of the masses and inversely as the squares of the distances apart*

It will be observed that the law of gravitation involves considerably more than can be derived from Kepler's laws of planetary motion, and it was by a master stroke of genius that Newton grasped it in its immense generality, and stated it so exactly that it has not been necessary to change a syllable in more than 200 years When contemplated in its entirety it is one of the grandest conceptions in the physical sciences

56 Examples of finding Law of Force (*a*) If a particle describes a circle passing through the origin, the law of force under which it moves is a very simple expression Let a represent the radius, then the polar equation of the circle is

$$r = 2a\cos\theta,$$

$$u = \frac{1}{2a\cos\theta}$$

Therefore

$$\frac{d^2u}{d\theta^2} + u = 8a^2u^3$$

Substituting in (25), it follows that

$$f = \frac{8a^2h^2}{r^5}$$

(*b*) Suppose the particle describes an ellipse with the origin at the center, it is required to find the law of central force The polar equation of an ellipse with the center as origin is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}$$

From this it follows that

$$\left\{\begin{aligned} bu &= \sqrt{1 - e^2\cos^2\theta}, \\ b\frac{d^2u}{d\theta^2} &= \frac{e^2\cos^2\theta - e^2\sin^2\theta}{\sqrt{1 - e^2\cos^2\theta}} - \frac{e^4\sin^2\theta\cos^2\theta}{(1 - e^2\cos^2\theta)^{\frac{3}{2}}}, \\ u + \frac{d^2u}{d\theta^2} &= \frac{1 - e^2}{b^4}\,\frac{1}{u^3} \end{aligned}\right.$$

Substituting in (25), the expression for the central force is found to be

$$f = \frac{h^2(1 - e^2)}{b^4}\, r = k^2 r$$

THE UNIVERSALITY OF NEWTON'S LAW

57 Double Star Orbits The law of gravitation has been *proved* so far to exist only in the solar system, the question whether it is actually *universal* naturally presents itself The fixed stars are so remote that it is not possible to observe planets revolving around them, if indeed they have such attendants The only observations thus far obtained which throw any light upon the subject are those of the motions of the double stars

Double star astronomy started about 1780 with the search for close stars by Sir William Herschel for the purpose of determining parallax by the differential method A few years were sufficient to show him, to his great surprise, that in some cases the two components of a pair were revolving around each other, and that, therefore, they were physically connected as well as being apparently in the same part of the sky The discovery and measurement of these systems has been pursued with increasing interest and zeal by astronomers until the catalogues now contain many thousands of these objects The relative motions are so slow in most cases that only a few have yet completed one revolution, or enough of one revolution so that the shapes of their orbits are known with certainty There are now about thirty pairs whose observed angular motions have been sufficiently great to prove, within the errors of the observations, that they move in ellipses with respect to each other in such a manner that the law of areas is fulfilled But in no case is the primary at the focus, or at the center, of the relative ellipse described by the companion, but occupies some other different place within the ellipse, the position varying greatly in different systems

From the observations and the converse of the law of areas it follows that the resultant of the forces acting upon one star of a pair is always directed toward the other The law of variation of the intensity of the force depends upon the position in the ellipse which the center of force occupies It must not be overlooked at this point that the orbits of the stars are not observed directly, but that it is their projections upon the planes tangent to the celestial sphere at their respective places which are seen The effect of this sort of projection is to change the true ellipse into a different apparent ellipse whose major axis has a different direction, and one that is differently situated with respect to the central star, indeed, it might happen that if one of the stars was really in the focus of the true ellipse described by the other, the projection would be such as to make it lie on the minor axis of the apparent ellipse

Astronomers have assumed that the apparent departure of the central star from the focus of the ellipse described by the companion is due to projection, and have then computed the angle of the line of nodes and the inclination No inconsistencies are introduced, but the

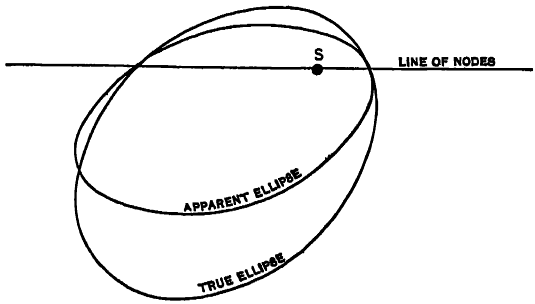

Fig 9

possibility remains that the assumptions are not true The question of what the law of force must be if it is not Newton's law of gravitation will now be investigated

58 Derivation of Law of Force If the force varied directly as the distance the primary star would be at the center of the ellipse described by the secondary (Art 53) No projection would change this relative position, and since such a condition has never been observed it is inferred that the force does not vary directly as the distance

The condition will now be imposed that the curve shall be a conic with general position for the origin, and the law of central force will be found The equation of the general conic is

$$ax^2 + 2bxy + cy^2 + 2dx + 2fy = g \tag{26}$$

Transforming to polar coordinates and putting $r = \frac{1}{u}$, this equation gives

$$u = A \sin \theta + B \cos \theta \pm \sqrt{C \sin 2\theta + D \cos 2\theta + H}, \tag{27}$$

where

$$\begin{cases} A = \dfrac{f}{g}, \quad B = \dfrac{d}{g}, \quad C = \dfrac{fd + bg}{g^2}, \quad D = \dfrac{d^2 + ag - f^2 - cg}{2g^2}, \\ H = \dfrac{d^2 + ag + f^2 + cg}{2g^2} \end{cases}$$

Differentiating (27) twice, it is found that

$$\frac{d^2u}{d\theta^2} = -A \sin \theta - B \cos \theta \pm \frac{-C^2 - D^2 - (C \sin 2\theta + D \cos 2\theta)^2 - 2H(C \sin 2\theta + D \cos 2\theta)}{(C \sin 2\theta + D \cos 2\theta + H)^{\frac{3}{2}}} \tag{28}$$

Substituting (27) and (28) in (25), it is found that

$$(29) \qquad f = \pm \frac{h^2}{r^2} \frac{(H^2 - C^2 - D\ )}{(C \sin 2\theta + D \cos 2\theta + H)^{\frac{3}{2}}}$$

This becomes as a consequence of (27)

$$(30) \qquad f = \pm \frac{h^2}{r} \frac{(H^2 - C^2 - D^2)}{\left(\frac{1}{r} - A \sin \theta - B \cos \theta\right)^3}$$

These two equations are the general expressions for the law of central force when the particle describes a conic section. The force depends, except in special cases, upon both the distance of the particle and its direction from the origin.

It does not seem reasonable to suppose that the attraction of two stars for each other depends upon their orientation in space. Equation (29) becomes independent of θ if $C = D = 0$, and (30), if $A = B = 0$. The first gives

$$f = \pm \frac{\text{constant}}{r^2},$$

and the second,

$$f = \pm \text{constant } r.$$

The first is Newton's law, and the second is excluded by the fact that no primary star has been found in the center of the orbit described by the companion. Hence, if the attraction does not depend upon the direction of the bodies from each other, Newton's law of gravitation operates in the binary systems.

59. Geometrical Interpretation of the Second Law. The general expression for the central force given in (30) may be put in a very simple and interesting form. Let $g^3 h^2 (H^2 - C^2 - D^2) = N$, and transform $\left(\frac{1}{r} - A \sin \theta - B \cos \theta\right)$ into rectangular coordinates and the original constants; then (30) becomes

$$(31) \qquad f = \frac{\mp Nr}{(dx + fy - g)^3}$$

The equation of the polar of the point (x', y') with respect to the general conic (26) is*

$$axx' + b(xy' + yx') + cyy' + d(x + x') + f(y + y') - g = 0$$

When (x', y') is the origin this equation becomes

$$(32) \qquad dx + fy - g = 0,$$

* Salmon's *Conic Sections*, Art. 89.

and has the same form as the denominator of (31) The values of x and y in (31) are such that they satisfy the equation of the conic, while those in (32) satisfy the equation of the polar line They are, therefore, in general numerically different But the distance from any point (x, y) of the conic to the polar line is given by the equation

$$p = \frac{dx + fy - g}{\sqrt{d^2 + f^2}},$$

where x and y are the coordinates of points on the conic Let

$$N' = \frac{N}{(d^2 + f^2)^{\frac{3}{2}}},$$

then (31) becomes

$$f = \mp \frac{N' r}{p^3} \tag{33}$$

Therefore, *if a particle moving subject to a central force describes any conic, the intensity of the force varies directly as the distance of the particle from the origin, and inversely as the cube of its distance from the polar of the origin with respect to the conic*

60 Examples (*a*) When the orbit is a central conic with the origin at the center, the polar line recedes to infinity, and $\frac{N'}{p^3}$ must be regarded as a constant Then the force varies directly as the distance as was shown in Art 56 (*b*)

(*b*) When the origin is at one of the foci of the conic the polar line is the directrix, and $p = \frac{r}{e}$, where e is the eccentricity Then (33) becomes

$$f = \mp \frac{N' e^3}{r^2}$$

This is Newton's law and was derived from the same conditions in Art 55

VIII PROBLEMS

1 Find the vis viva integral when $f = \frac{c}{r^2}$, $f = cr$, $f = \frac{c}{r^n}$

2 Suppose that in Art 53 the particle is projected orthogonally from the x axis, find the equations corresponding to (19) and (20) Suppose still further that $k = 1$, $x_0 = 1$, find the initial velocity such that the eccentricity of the ellipse may be 1/2

$$Ans \begin{cases} v_0 = \frac{\sqrt{3}}{2}, \\ \text{or} \\ v_0 = \frac{3}{2}\sqrt{3} \end{cases}$$

3 Derive (25) starting from the equation $y=r\sin\theta$ Express (25) in terms of r and θ

4 Find the central force under which a particle may describe the spiral $r=\dfrac{1}{c\theta}$, the spiral $r=e^{\theta}$

$$Ans \quad f=\frac{h^2}{r^3}, \qquad f=\frac{2h^2}{r^3}$$

5 Find the central force under which a particle may describe the lemniscate $r^2=a^2\cos 2\theta$

$$Ans \quad f=\frac{3h^2a^4}{r^7}$$

6 Find the central force under which a particle may describe the cardioid $r=a(1+\cos\theta)$

$$Ans \quad f=\frac{3ah^2}{r^4}$$

7 Suppose the particle describes an ellipse with the origin in its interior, at a distance n from the x axis and m from the y axis (a) Show that the laws of force are

$$\begin{cases} f=\dfrac{h}{r^2}\,\dfrac{(ac)^{\frac{1}{2}}}{[2mn\sin\theta\cos\theta+(a-c-n^2+m^2)\cos^2\theta+c-m^2]^{\frac{3}{2}}}, \\[2ex] f=\dfrac{h^2a^2c^2r}{[ac-am^2-cn^2-cny-amx]^3}, \end{cases}$$

where a and c have the same meaning as in (26), and where the polar axis is parallel to the major axis of the ellipse (b) If the origin is between the center and the focus show that the force at unit distance is a maximum for $\theta=0$ and is a minimum for $\theta=\dfrac{\pi}{2}$, that if the origin is between a focus and the nearest apse the maximum is for $\theta=\dfrac{\pi}{2}$ and the minimum for $\theta=0$, and that if the origin is on the minor axis the maximum is for $\theta=0$, and the minimum for $\theta=\dfrac{\pi}{2}$

8 Interpret equation (29) geometrically

Hint $\qquad C\sin 2\theta+D\cos 2\theta+H=\dfrac{(dx+fy)^2+g(ax^2+cy^2+2bxy)}{g^2r^2}$

The numerator of this expression set equal to zero is the equation of the tangents (real or imaginary) from the origin to the conic (Salmon's *Conic Sections*, Art 92)

9 Find the expressions for the central force when the orbit is an ellipse with the origin at an end of the major and minor axes respectively Show that they reduce to $\dfrac{k^2}{r^5}$ when the ellipse becomes a circle

$$Ans \quad \begin{cases} f=\dfrac{h\sqrt{c}}{ar}\,\dfrac{1}{\cos^3\theta}, \\[2ex] f=\dfrac{h^2\sqrt{a}}{cr^2}\,\dfrac{1}{\sin^3\theta} \end{cases}$$

Determination of the Orbit from the Law of Force

61 Force Varying as the Distance The problem of finding the orbit when the law of force is given is generally more difficult than the converse, since it involves the integration of (25) The method of integration varies with the different laws of force, and the character of the integrals depends upon the initial conditions The process will be illustrated first in the case in which the force varies as the distance, a problem treated by another method in Art 53

If $f=k^2r$, equation (25) becomes

$$\frac{k^2}{u}=h^2u^2\left(u+\frac{d^2u}{d\theta^2}\right),$$

or

$$\frac{d^2u}{d\theta^2}=\frac{k^2}{h^2}\frac{1}{u^3}-u$$

The integral of this equation is

$$\left(\frac{du}{d\theta}\right)^2=-\frac{k^2}{h^2}\frac{1}{u^2}-u^2+c_1,$$

whence

$$d\theta=\frac{\pm\, u\,du}{\left[\frac{c_1^2}{4}-\frac{k^2}{h^2}-\left(\frac{c_1}{2}-u^2\right)^2\right]^{\frac{1}{2}}} \tag{34}$$

Let

$$\frac{c_1}{2}-u^2=z, \qquad \frac{c_1^2}{4}-\frac{k^2}{h^2}=A^2$$

The constant A^2 must be positive in order that $\frac{du}{d\theta}$ may be real, as it is if the particle is started with real initial conditions

Using the upper sign, equation (34) becomes

$$2d\theta=\frac{-dz}{\sqrt{A^2-z^2}} \tag{35}$$

It is easily verified that the same final equation would be reached, when the initial conditions are substituted, if the lower sign were used The integral of (35) is

$$\cos^{-1}\frac{z}{A}=2(\theta+c_2),$$

or

$$z=A\cos 2(\theta+c_2)$$

Changing to the variable r, this equation becomes

$$r^2=\frac{2}{c_1-2A\cos 2(\theta+c_2)} \tag{36}$$

This is the polar equation of an ellipse with the origin at the center. Hence, a particle moving subject to an attractive force varying directly as the distance describes an ellipse with the origin at the center. The only exceptions are when the particle passes through the origin, and when it describes a circle. In the first $h = 0$, and equation (25) ceases to be valid; in the second c_1 has such a value that it satisfies the equation

$$\left(\frac{du}{d\theta}\right)_0 = -\frac{k^2}{h}\,\frac{1}{u_0^2} - u_0^2 + c_1 = 0,$$

and the equation of the orbit becomes $u = u_0$. In this case equation (34) fails.

62. Force Varying Inversely as the Square of the Distance. A particle is moving under the influence of a central attraction the intensity of which varies inversely as the square of the distance; it is required to determine its orbit when it is projected in any manner. Equation (25) is in this case

$$\frac{d^2 u}{d\theta^2} = \frac{k^2}{h^2} - u \tag{37}$$

This becomes, after integrating and solving so as to separate the variables,

$$\frac{\pm\, du}{\sqrt{c_1 + \frac{k^4}{h^4} - \left(\frac{k^2}{h^2} - u\right)^2}} = d\theta$$

The integrals of these equations are

$$\begin{cases} \sin^{-1} \dfrac{\left(\frac{k^2}{h^2} - u\right)}{\sqrt{c_1 + \frac{k^4}{h^4}}} = \theta + c, \\[2ex] \cos^{-1} \dfrac{\left(\frac{k^2}{h^2} - u\right)}{\sqrt{c_1 + \frac{k^4}{h^4}}} = \theta + c' \end{cases}$$

Solving for u and taking the reciprocal, it is found that

$$r = \frac{1}{\frac{k^2}{h^2} - \sqrt{c_1 + \frac{k^4}{h^4}}\,\sin(\theta + c)} = \frac{1}{\frac{k^2}{h^2} - \sqrt{c_1 + \frac{k^4}{h^4}}\,\cos(\theta + c_2')} \tag{38}$$

These are the polar equations of a conic section with the origin at the focus. When the constants c and c_2 are determined from the initial conditions they reduce to the same expression.

From this investigation and that of Art 55 it follows that, if the orbit is a conic section with the origin at one of the foci, then the body moves subject to a central force varying inversely as the square of the distance, and conversely, if the force varies inversely as the square of the distance, then the body will describe a conic section with the origin at one of the foci

Let p represent the parameter of the conic and e its eccentricity Then, comparing (38) with the ordinary polar equation of the conic, $r = \frac{p}{1 + e\cos\phi}$, it is found that

$$(39) \qquad \begin{cases} p = \dfrac{h^2}{k^2}, \\ e = \dfrac{h^2}{k^2}\sqrt{c_1 + \dfrac{k^4}{h^4}}, \end{cases}$$

and that $c_2 = \frac{3\pi}{2} - \omega$, where ω is the angle between the polar axis and the end of the major axis directed to the nearest apse The constants h^2 and c_1 are determined by the initial conditions, and they in turn define p and e by (39) If $e < 1$, the conic is an ellipse, if $e = 1$, the conic is a parabola, if $e > 1$, the conic is a hyperbola If $e = 0$, the conic is a circle, but it cannot take this value as defined by (39), for then the equation following (37) is not valid This case arises when the values of c_1, h, and u_0 are such that $c_1 + \frac{k^4}{h^4} - \left(\frac{k^2}{h^2} - u_0\right)^2 = 0$ Then u equals a constant, and the curve is a circle

63 Force Varying Inversely as the Fifth Power of the Distance In this case $f = \frac{k}{r^5}$, and (25) becomes

$$(40) \qquad k^2 u^5 = h^2 u \left(u + \frac{d^2u}{d\theta^2}\right)$$

Solving for $\frac{d^2u}{d\theta^2}$ and integrating, it is found that

$$(41) \qquad \left(\frac{du}{d\theta}\right)^2 = \frac{1}{2}\frac{k^2}{h^2}u^4 - u^2 + c_1$$

Therefore

$$(42) \qquad d\theta = \frac{du}{\sqrt{c_1 + \frac{1}{2}\frac{k^2}{h}u^4 - u^2}}$$

This cannot in general be integrated in terms of the elementary functions, but can be transformed into an elliptic integral of the first kind Then u, and consequently r, is expressible in terms of θ by

elliptic functions, and the orbits are spirals, their particular character depending upon the initial conditions

There are certain special cases which are worthy of notice

(*a*) If such a constant value of u be taken that it fulfills the right member of (41) when it is set equal to zero, then r is a constant and the orbit is a circle with the origin at the center It is easily seen that a similar special case will occur for a central force varying as any power of the distance

(*b*) Another special case is that in which the initial conditions are such that $c_1 \neq 0$ and the right member of (41) is a perfect square That is, $c_1 = \frac{h}{2k^2}$ Then equation (41) becomes

$$\left(\frac{du}{d\theta}\right) = \left(\frac{1}{\sqrt{2}}\frac{k}{h}u - \frac{1}{\sqrt{2}}\frac{h}{k}\right)^2 = \frac{1}{2}\left(A^2 u^2 - \frac{1}{A^2}\right)^2$$

The integral of this equation is

$$\log \frac{1 + A^2 u}{1 - A^2 u} = \sqrt{2}\,(\pm\theta - c_2)\,,$$

whence

$$r = -A^2 \frac{[1 + e^{\sqrt{2}(\pm\theta - c_2)}]}{[1 - e^{\sqrt{2}(\pm\theta - c_2)}]} \tag{43}$$

This is the equation of the hyperbolic cotangent*

(*c*) If the initial conditions are such that $c_1 = 0$, equation (41) gives

$$\pm\, d\theta = \frac{du}{u\sqrt{\frac{1}{2}\frac{k}{h^2}u^2 - 1}},$$

the integral of which is

$$\pm\,\theta = \cos^{-1}\left(\frac{\sqrt{2}h}{ku}\right) + c_2$$

Taking the cosines of both members and solving for r, the polar equation of the orbit is found to be

$$r = \frac{k}{\sqrt{2}h}\cos(c \mp \theta), \tag{44}$$

which is the equation of a circle with the origin on the circumference

(*d*) If none of these conditions is fulfilled the right member of (41) is a biquadratic, and equation (42) may be written in the form

$$\pm\, d\theta = \frac{C\,du}{\sqrt{\pm(1 \pm \alpha^2 u^2)(1 \pm \beta^2 u^2)}}, \tag{45}$$

* See Byerly's *Integral Calculus* chap II

where C, α^2, and β^2 are constants which depend upon the coefficients of (41) in a simple manner Equation (45) leads to an elliptic integral which expresses θ in terms of u Taking the inverse functions and the reciprocals, r is expressed as an elliptic function of θ The curves are spirals of which the circle through the origin, and the one around the origin as center, are limiting cases

If the curve is a circle through the origin the force varies inversely as the fifth power of the distance (Art 56), but if the force varies inversely as the fifth power of the distance the curves which the particle will describe are spirals of which the circle is a particular limiting case On the other hand, if the curve is a conic with the origin at the center or at one of the foci, the force varies directly as the distance, or inversely as the square of the distance, and conversely, if the force varies directly as the distance, or inversely as the square of the distance, the orbits are always conics with the origin at the center, or at one of the foci respectively [Arts 53, 55, 56 (*b*)] A complete investigation is necessary for every law to show this converse relationship

IX PROBLEMS

1 Discuss the motion of the particle by the general method for linear equations when the force varies inversely as the cube of the distance Trace the curves in the various special cases

2 When the force is $f=\frac{\mu}{r^2}+\frac{\nu}{r^3}$ show that, if $\nu<h$, the general equation of the orbit described has the form

$$r=\frac{a}{1-e\cos(k\theta)},$$

where a, e, and k are the constants depending upon the initial conditions and μ and ν Observe that this may be regarded as being a conic section whose major axis revolves around the focus with uniform angular velocity

$$n=(1-k)\frac{2\pi}{T},$$

where T is the period of revolution

3 Show that in the case of a central force the motion along the radius vector is defined by the equation

$$\frac{d^2r}{dt^2}=-f-\frac{h^2}{r^3}$$

4 Express C, α^2, and β^2 of equation (45) in terms of the initial conditions For original projections at right angles to the radius vector investigate all the possible cases, reducing the integrals to the normal form, and expressing r as elliptic functions of θ Draw the curves in each case

5 Suppose the law of force is that given in (29), i e

$$f=\frac{M}{r^2(C\sin 2\theta+D\cos 2\theta+H)^{\frac{3}{2}}}=\frac{M}{r^2[\phi(\theta)]^{\frac{3}{2}}}$$

Integrate the differential equation of the orbit, (25), by the method of variation of parameters, and show that the general solution has the form

$$\frac{1}{r}=c_1\cos\theta+c_2\sin\theta+\sqrt{\phi(\theta)},$$

where c_1 and c_2 are constants of integration Show that the curve is a conic

6 Suppose the law of force is that given by (30), i e

$$f=\frac{N}{r\left(\frac{1}{r}-A\sin\theta-B\cos\theta\right)^3}$$

Substitute in (25) and derive the general equation of the orbit described

Hint Let $u=v+A\sin\theta+B\cos\theta$, then (25) becomes

$$\frac{d^2v}{d\theta^2}+v=\frac{Nh^{-2}}{v^3}$$

Ans $$\frac{1}{r}=A\sin\theta+B\cos\theta+\sqrt{c_1\cos^2\theta+c_2\sin 2\theta+c_3\sin^2\theta},$$

which is the equation of a conic section

7 Suppose the law of force is

$$f=\frac{c_1+c_2\cos 2\theta}{r^2},$$

show that, for all initial projections, the orbit is an algebraic curve of the fourth degree unless $c_2=0$, when it reduces to a conic

HISTORICAL SKETCH AND BIBLIOGRAPHY

The subject of central forces was first discussed by Newton. In Sections II and III of the First Book of the *Principia* he gave a splendid geometrical treatment of the subject, and arrived at some very general theorems. These portions of the *Principia* especially deserve careful study.

All the simpler cases were worked out in the eighteenth century by analytical methods. A few examples are given in detail in Legendre's *Traité des Fonctions Elliptiques*. An exposition of principles and a list of examples are given in nearly every work on analytical mechanics; among the best of these treatments are the Fifth Chapter in Tait and Steele's *Dynamics of a Particle*, and the Tenth Chapter, vol. I, of Appell's *Mécanique Rationelle*. Stader's memoir, *De orbitis et motibus puncti cuiusdam corporei circa centrum attractionum aliis, quam Newtonia, attractionis legibus sollicitati*, vol. XLVI, *Journal fur Mathematik*, treats the subject in great detail.

The problem of finding the general expression for the possible laws of force operating in the binary star systems was proposed by M. Bertrand in vol. LXXXIV of the *Comptes Rendus*, and was immediately solved by MM. Darboux and Halphen, and published in the same volume. The treatment given above in the text is similar to that given by M. Darboux, which has also been reproduced in a note at the end of the *Mécanique* of M. Despeyrous. The method of M. Halphen is given in Tisserand's *Mécanique Céleste*, vol. I, p. 36, and in Appell's *Mécanique Rationelle*, vol. I, p. 372. It seems to have been generally overlooked that Newton had treated the same problem in the *Principia*, Book I, Scholium to Proposition XVII. This was reproduced and shown to be equivalent to the work of MM. Darboux and Halphen by Professor Glaisher in the *Monthly Notices of R. A. S.*, vol. XXXIX.

M. Bertrand has shown (*Comptes Rendus*, vol. LXXVII.) that the only laws of central force under the action of which a particle will describe a conic section for all initial conditions, provided the initial velocity is not too great, are $f = \pm \frac{k^2}{r^2}$ and $f = \pm k^2 r$. M. Koenigs has shown (*Bulletin de la Société Mathématique*, vol. XVII.) that the only laws of central force depending upon the distance alone, for which the curves described are algebraic for all initial conditions, are $f = \pm \frac{k^2}{r^2}$ and $f = \pm k^2 r$.

CHAPTER IV

THE POTENTIAL AND ATTRACTIONS OF BODIES

64 THE previous chapters have been concerned with problems in which the law of force was given, or with the discovery of the law of force when the orbits were given All the investigations were made as though the masses were mere points instead of being of finite size When forces exist between every two particles of all the masses involved, bodies of finite size cannot be assumed to attract like particles This leads to the problem of determining the way in which finite bodies of different shapes attract each other

It follows from Kepler's laws and the principles of central forces that, if the planets are regarded as being of infinitesimal dimensions compared to their distances from the sun, they move under the influence of forces which are directed to the center of the sun and which vary inversely as the squares of the distances (Art 55) This suggests the idea that the law of inverse squares may account for the motions still more exactly if the bodies are regarded as being of finite size, with every particle attracting every other particle in the system The appropriate investigation shows that this is true

This chapter will be devoted to an exposition of general methods of finding the attractions of bodies of any shape on unit particles in any position, exterior or interior, when the forces vary inversely as the squares of the distances The astronomical applications will be to the attractions of spheres and oblate spheroids, to the variation in the surface gravitation of the planets, and to the perturbations of the motions of the satellites due to the flattening of the planets

65 **Solid Angles** If a straight line constantly passing through a fixed point is moved until it takes its original position, it generates a conical surface of two sheets whose vertices are at the given point The area which one end of this double cone cuts out of the surface of

the unit sphere whose center is at the given point will be called the *solid angle* of the cone, or, the area cut out of any concentric sphere divided by the square of its radius measures the solid angle

Since the area of a spherical surface equals the product of 4π and the square of its radius, it follows that the sum of all the solid angles about a point is 4π The sum of the solid angles of one half of all the double cones which can be constructed about a point without intersecting each other is 2π

The volume contained within an infinitesimal cone whose solid angle is ω and between two spherical surfaces whose centers are at the vertex of the cone, approaches as a limit the product of the solid angle, the square of the distance of the spherical surfaces from the vertex, and the distance between them, as they approach each other If the centers of the spherical surfaces are at a point not in the axis of the cone the volume approaches as a limit the product of the solid angle, the square of the distance from the vertex, the distance between the spherical surfaces, and the reciprocal of the cosine of the angle between the axis of the cone and the radius from the center of the sphere, or, it is the product of the solid angle, the square of the distance from the vertex, and the intercept on the cone between the spherical surfaces Thus the volume of $abdc$ is $V = \omega\,\overline{aO}^2\ ab$

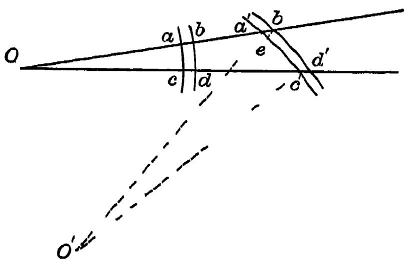

Fig 10

The volume of $a'b'd'c'$ is

$$V' = \frac{\omega\,\overline{a'O}^2\ b'e'}{\cos\,(Oa'O')} = \omega\,\overline{a'O}^2\ a'b'$$

Sometimes it will be convenient to use one of these expressions and sometimes the other

66 The Attraction of a Thin Homogeneous Spherical Shell upon a Particle in its Interior The attractions of spheres and other simple figures were treated by Newton in the *Principia*,

Book I, Section 12 The following demonstration is essentially as given by him

Consider the spherical shell contained between the infinitely near spherical surfaces S and S', and let P be a particle of unit mass situated within it Construct an infinitesimal cone whose solid angle

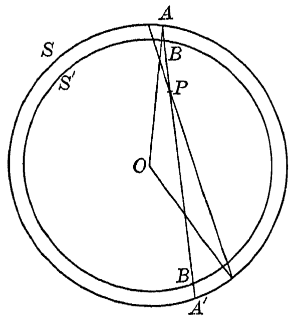

Fig 11

is ω with its vertex at P Let σ be the density of the shell Then the mass of the element of the shell at A is $m = \sigma \overline{AB}\, \omega \overline{AP}^2$, likewise the mass of the element of A' is $m' = \sigma \overline{A'B'}\, \omega \overline{A'P}$ The attractions of m and m upon P are respectively

$$\alpha = \frac{k\, m}{\overline{AP}^2}, \qquad \alpha' = \frac{k\, m'}{\overline{A'P}^2}$$

Since $\overline{A'B'} = \overline{AB}$, $\alpha = k^2 \overline{AB}\, \omega\sigma = \alpha'$ This holds for every infinitesimal solid angle with vertex at P, *therefore a thin homogeneous spherical shell attracts particles within it equally in opposite directions*

This holds for any number of thin spherical shells and, therefore, for shells of finite thickness

67 The Attraction of a Thin Homogeneous Ellipsoidal Shell upon a Particle in its Interior The theorem of this article is given in the *Principia*, Book I, Prop XCI, Cor 3

Let a *homoeoid* be defined as a thin shell contained between two similar surfaces similarly placed Thus, an *elliptic homoeoid* is a thin shell contained between two similar ellipsoids similarly placed

Consider the attraction of the elliptic homoeoid whose surfaces are the ellipsoids E and E' upon the interior unit particle P Construct an infinitesimal cone whose solid angle is ω with vertex at P The masses of the two infinitesimal elements at A and A' are respectively

$m = \sigma \overline{AB}\,\omega\,\overline{AP}^2$ and $m' = \sigma \overline{A'B'}\,\omega\,\overline{A'P}^2$ The attractions are $\alpha = \dfrac{k^2 m}{\overline{AP}^2}$ and $\alpha' = \dfrac{k^2 m'}{\overline{A'P}^2}$ Construct a diameter $\overline{CC'}$ parallel to $\overline{AA'}$ in the elliptical section of a plane through the cone, and draw its conjugate $\overline{DD'}$ They are conjugate diameters in both elliptical sections, E and E', therefore $\overline{DD'}$ bisects every chord parallel to $\overline{CC'}$, and $\overline{AB} = \overline{A'B'}$

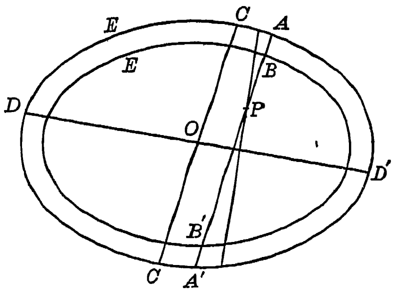

Fig 12

The attractions of the elements at A and A' upon P are therefore equal This holds for every infinitesimal solid angle whose vertex is at P, therefore *the attractions of a thin elliptic homoeoid upon an interior particle are equal in opposite directions*

This holds for any number of thin shells and, therefore, for shells of finite thickness

68 The Attraction of a Thin Homogeneous Spherical Shell upon an Exterior Particle Newton's Method Let $AHKB$ and $ahkb$ be two equal thin spherical shells with centers at O

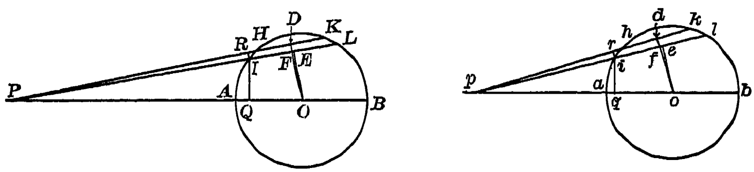

Fig 13

and o respectively Let two unit particles be placed at P and p, unequal distances from the centers of the shells Draw any secants from p cutting off the arcs il and hk, and let the angle kpl approach

zero as a limit. Draw from P the secants PL and PK, cutting off the arcs IL and HK equal respectively to il and hk. Draw oe perpendicular to pl, od perpendicular to pk, iq perpendicular to pb, and ir perpendicular to pk. Draw the corresponding lines in the other figure. Then it follows from similar triangles and the ratios at the limits that

$$\begin{cases} PI : PF = RI : DF, \\ \quad pf : pi = DF\,(=df) : ri. \end{cases}$$

The product of these two proportions is

(1) $$PI\ pf : PF\ pi = RI : ri = HI : hi.$$

From the similar triangles PIQ and POE, it follows that

$$PI : PO = IQ : OE,$$

and similarly

$$po : pi = OE\,(=oe) : iq.$$

The product of these two proportions is

(2) $$PI\ po : PO\ pi = IQ : iq.$$

Taking the products of (1) and (2), it follows that

(3) $$\overline{PI}\ pf\ po : \overline{pi}^2\ PF\ PO = HI\ IQ : hi\ iq.$$

Rotate the figures around the diameters PB and pb, and call the masses of the circular rings generated by HI and hi, M and m respectively; then

(4) $$HI\ IQ : hi\ iq = M : m.$$

Let the attractions of M and m upon P and p in the directions I and i be A_I and α_i respectively. Therefore

(5) $$A_I : \alpha_i = \frac{M}{\overline{PI}^2} : \frac{m}{\overline{pi}^2}.$$

Let the forces A_I and α_i be resolved into components along and perpendicular to the lines PB and pb. Since the figures have been rotated around these lines as axes, the components which are perpendicular to them will balance each other. Let A_O and α_o represent the components toward the centers; then

$$A_O : \alpha_o = A_I\,\frac{PQ}{PI} : \alpha_i\,\frac{pq}{pi} = A_I\,\frac{PF}{PO} : \alpha_i\,\frac{pf}{po}.$$

This proportion becomes as a consequence of (3), (4), and (5)

(6) $$A_O : \alpha_o = \overline{po}^2 : \overline{PO}^2.$$

Therefore, the circular rings attract the exterior particles toward the centers of the shells with forces which are inversely proportional to the squares of their respective distances from these centers In a similar manner the same may be proved for the rings KL and kl

Now let the lines PK and pk vary from coincidence with the diameters PB and pb to tangency with the spherical shells The results are true at every position separately, and hence for all at once Therefore, *the resultants of the attractions of thin spherical shells upon exterior particles are directed toward their centers, and the intensities of the forces vary inversely as the squares of the distances of the particles from the centers*

If the body is a homogeneous sphere, or is made up of homogeneous spherical layers, the theorem will hold for each one separately, and consequently for all of them combined

69 Comments upon Newton's Method While the demonstration above is given in the language of Geometry it really depends upon the principles which are fundamental in the Calculus Letting the angle kpl approach zero as a limit is equivalent to taking a differential element, the rotation around the diameters is equivalent to an integration with respect to one of the polar angles, the variation of the line pk from coincidence with the diameter to tangency is equivalent to an integration with respect to the other polar angle, and the summation of the infinitely thin shells to form a solid sphere is equivalent to an integration with respect to the radius

Since the work has been done entirely in ratios the results obtained do not relate to the absolute intensity of the attraction This is of scarcely less importance than a knowledge of the manner in which it varies with the distance

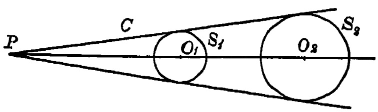

Fig 14

Take two equally dense spherical shells, S_1 and S_2, internally tangent to the cone C Let $PO_1 = a_1$, $PO_2 = a_2$, and M_1 and M_2 be the masses of S_1 and S_2 respectively The two shells attract the particle P equally, for, any solid angle which includes part of one shell also includes a

similar part of the other The masses of these included parts are as the squares of their distances, and their attractions are inversely as the squares of their distances, whence the equality of their attractions upon P Let A represent the common attraction, then remove S_1 so that its center is at O_2 Let A' represent the intensity of the attraction of S_1 in the new position, then, by the theorem of Art 68,

$$\frac{A'}{A}=\frac{a_1^2}{a^2}=\frac{M_1}{M_2}$$

Therefore, *the two shells attract a particle at the same distance with forces directly proportional to their masses* From this and the previous theorem, it follows that *a particle exterior to a sphere which is homogeneous in concentric layers is attracted toward its center with a force directly proportional to the mass of the sphere and inversely as the square of the distance from its center, or, as though the mass of the sphere were all at its center*

Since the heavenly bodies nearly fulfill these conditions they may be regarded as material points in the discussion of their mutual interactions except when they are relatively near each other as in the case of the planets and their respective satellites

70 The Attraction of a Thin Homogeneous Spherical Shell upon an Exterior Particle Thomson and Tait's Method Let O be the center of the spherical shell whose radius is a, and whose thickness is Δa, P the position of the attracted particle, and PO a line

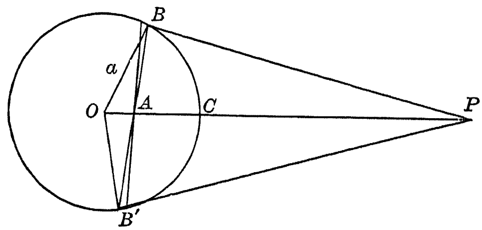

Fig 15

from the attracted particle to the center cutting the spherical surface in C Take the point A so that $PO \quad OC = OC \quad OA$, and construct the infinitesimal cone whose solid angle is ω with its vertex at A Let σ be the density of the shell Then the elements of mass at B and B' are respectively

$$m=\sigma\omega \overline{AB}^2\frac{\Delta a}{\cos(OBA)},\quad m'=\sigma\omega \overline{AB'}^2\frac{\Delta a}{\cos(OB'A)}$$

The attractions of the two masses upon P are respectively

$$(7)\qquad \begin{cases} \alpha = k^2\sigma\omega\dfrac{\overline{AB}^2}{\overline{BP}^2}\ \dfrac{\Delta a}{\cos(OBA)}, \\ \alpha' = k^2\sigma\omega\dfrac{\overline{AB'}^2}{\overline{B'P}^2}\ \dfrac{\Delta a}{\cos(OB'A)} \end{cases}$$

From the construction of A

$$PO\ \ OC = OC\ \ OA,$$

or

$$PO\ \ OB = OB\ \ OA$$

Hence the triangles POB and BOA, having a common angle included between proportional sides, are similar Therefore

$$\frac{AB}{BP} = \frac{OB}{OP} = \frac{a}{OP}$$

Similarly

$$\frac{AB'}{B'P} = \frac{a}{OP}$$

The angle OBA equals the angle $OB'A$ Then equations (7) become

$$(8)\qquad \begin{cases} \alpha = k^2\sigma\omega\dfrac{a^2}{\overline{OP}^2}\ \dfrac{\Delta a}{\cos(OBA)}, \\ \alpha' = k^2\sigma\omega\dfrac{a^2}{\overline{OP}^2}\ \dfrac{\Delta a}{\cos(OBA)} \end{cases}$$

Hence $\alpha = \alpha'$

The angles BPO and $B'PO$ are respectively equal to OBA and $OB'A$, therefore they are equal to each other The resultant of the two equal attractions α and α' is in the line bisecting the angle between them, or in the direction of O, and is given in magnitude by the equation

$$\Delta R = \alpha\cos(BPO) + \alpha'\cos(B'PO) = 2\alpha\cos(OBA)$$

This becomes, as a consequence of (8),

$$\Delta R = 2k^2\sigma\omega\frac{a^2\Delta a}{\overline{OP}^2}$$

This equation is true for every solid angle with vertex at A and consequently for their sum Therefore the attraction of the whole

spherical shell upon the exterior particle is, summing with respect to ω,

$$R = 4\pi k^2 \sigma \frac{a^2 \Delta a}{\overline{OP}^2} = \frac{k\, M}{\overline{OP}^2},$$

or, it varies directly as the mass and inversely as the square of the distance of the particle from the center

71 The Attraction upon a Point in a Thin Homogeneous Spherical Shell Let O be the center of the spherical shell of thickness Δa, and P the position of the attracted particle Construct

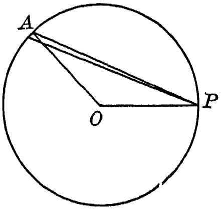

Fig 16

a cone whose solid angle is ω with its vertex at P Let σ be the density of the shell, then the mass of the section cut out at A by the cone is $\sigma\omega AP^2 \dfrac{\Delta a}{\cos(OAP)}$ The attraction of the element along AP is $\alpha = k^2 \sigma\omega \dfrac{\overline{AP}^2}{\overline{AP}^2} \dfrac{\Delta a}{\cos(OAP)}$ The resultant attraction of the shell is in the direction PO since the mass is symmetrically situated with respect to this line The component in the direction PO is

$$\Delta R = \alpha \cos(APO) = \alpha \cos(OAP) = k\ \sigma\omega\Delta a$$

The attraction of the whole shell is $R = 2k^2\sigma\pi\Delta a = \dfrac{k\, M}{2a^2}$

It follows from this equation and the results obtained in Arts 66 and 69 that the attraction on an interior particle infinitely near the shell is zero, on a particle in the shell, $\dfrac{k\, M}{2a^2}$, and on an exterior particle infinitely near the shell, $\dfrac{k^2 M}{a^2}$ *

* See note on the attraction of spherical shells, Lagrange, *Collected Works* vol VII p 591

X PROBLEMS

1 Suppose any two similar bodies are placed in perspective Show that a particle at their center of perspectivity is attracted inversely as their linear dimensions if they are thin rods of equal density, equally, if they are thin shells of equal density, and directly as their linear dimensions if they are solids of equal density

2 Prove that the attractions of two spheres of equal density for particles upon their surfaces are to each other as their radii

3 Prove that the attraction of a homogeneous sphere upon a particle in its interior varies directly as the distance of the particle from the center

4 Prove that all the frustums of equal height of any homogeneous cone attract a particle at its vertex equally

5 Find the law of density such that the attraction of a sphere for a particle upon its surface shall be independent of the size of the sphere

6 Prove that the attraction of a uniform thin rod, bent in the form of an arc of a circle, upon a particle at the center of the circle is the same as that which the mass of a similar rod equal to the chord joining the extremities would exert if it were concentrated at the middle point of the arc.

7 Prove that the attraction of a thin uniform straight rod is the same in magnitude and direction on an exterior particle as that of a circular arc of the same density, with its center at the particle, and subtending the same angle as the rod, and which is tangent to the rod

8 Prove that if straight uniform rods form a polygon all of whose sides are tangent to a circle, a particle at the center of the circle is attracted equally in opposite directions by the rods

9 Prove that two spheres attract each other as though their masses were all at their respective centers

72 The General Equations for the Components of Attraction and for the Potential when the Attracted Particle is not a Part of the Attracting Mass The geometrical methods of the preceding articles are special, being efficient only in the particular cases to which they are applied, the analytical methods which follow are characterized by their uniformity and generality, and illustrate again the advantages of processes of this nature

Consider the attraction of the finite mass M whose density is σ upon the unit particle P, which is not a part of it That is, P is exterior to M or within some cavity in it Let the coordinates of P

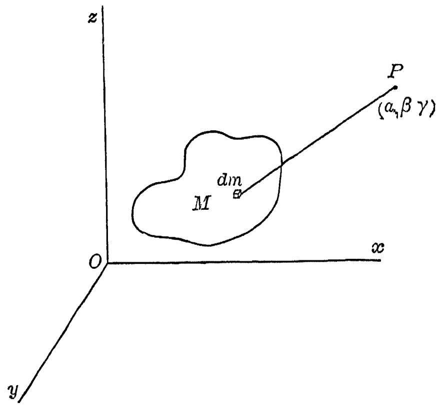

Fig 17

be α, β, γ Let its distance from any element, dm, of M be ρ Then the components of attraction parallel to the coordinate axes are respectively

$$
(9)\quad \begin{cases} X = -k^2 \displaystyle\int_{(M)} \frac{dm}{\rho}\;\frac{(\alpha - x)}{\rho} = -k^2 \int_{(M)} \frac{(\alpha - x)}{\rho^3}\,dm, \\[2ex] Y = -k^2 \displaystyle\int_{(M)} \frac{(\beta - y)}{\rho^3}\,dm, \\[2ex] Z = -k^2 \displaystyle\int_{(M)} \frac{(\gamma - z)}{\rho^3}\,dm, \end{cases}
$$

where
$$\rho^2 = (\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2,$$
$$\sigma = f(x, y, z)$$

$\int_{(M)}$ signifies that the integral must be extended over the whole mass M Then, if σ is a finite continuous function of the coordinates, as

will always be the case in what follows, X, Y, and Z are finite definite quantities In practice dm is expressed in terms of σ and the ordinary rectangular or polar coordinates, and X, Y, and Z are found by triple integrations

The three integrals (9) may be made to depend upon a single integral in a very simple manner Let

$$V = \int_{(M)} \frac{dm}{\rho} \tag{10}$$

V is called the *potential function*, the term having been introduced by Green in 1828 It will be spoken of as the potential of M upon P at the point (α, β, γ)

Since P is not a part of the mass M, ρ does not vanish in the region of integration The limits of the integral are independent of the position of the attracted particle, therefore the function under the integral sign may be differentiated* with respect to the parameters α, β, γ The partial derivatives are

$$\left\{\begin{aligned} \frac{\partial V}{\partial \alpha} &= -\int_{(M)} \frac{(\alpha - x)}{\rho^3}\, dm, \\ \frac{\partial V}{\partial \beta} &= -\int_{(M)} \frac{(\beta - y)}{\rho^3}\, dm, \\ \frac{\partial V}{\partial \gamma} &= -\int_{(M)} \frac{(\gamma - z)}{\rho^3}\, dm \end{aligned}\right.$$

Comparing these equations with (9), it is found that

$$\left\{\begin{aligned} X &= k^2 \frac{\partial V}{\partial \alpha}, \\ Y &= k^2 \frac{\partial V}{\partial \beta}, \\ Z &= k^2 \frac{\partial V}{\partial \gamma} \end{aligned}\right. \tag{11}$$

Therefore, in the case in which P is *not* a part of M the solution of the problem of finding the components of attraction depends upon the computation of the single function V

73 Case where the Attracted Particle is a Part of the Attracting Mass It will now be proved that the components of attraction and the potential have finite, definite values when the particle *is* a part of the attracting mass, and that in this case equations (11) hold also

* Byerly's *Integral Calculus*, Art 89

Consider the mass M and the particle P constituting a part of it Construct around P as a center a small sphere with radius ϵ Let the components of attraction and the potential of the mass exterior to

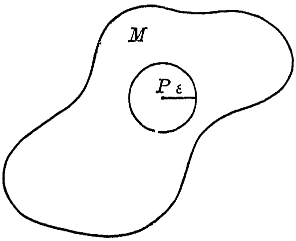

Fig 18

the small sphere upon the particle P be X', Y', and Z', and V' respectively The components of attraction and the potential of the whole mass upon P are then

$$\begin{cases} X = \lim\limits_{\epsilon=0} X', \\ Y = \lim\limits_{\epsilon=0} Y', \\ Z = \lim\limits_{\epsilon=0} Z', \\ V = \lim\limits_{\epsilon=0} V' \end{cases}$$

Let M' represent the mass exterior to the small sphere, then

$$(12) \qquad \begin{cases} X' = -k^2 \displaystyle\int_{(M')} \frac{(a-x)}{\rho^3}\, dm, \\ V' = \displaystyle\int_{(M')} \frac{dm}{\rho} \end{cases}$$

If the origin is taken at P the expression for the element of mass in polar coordinates is

$$dm = \sigma\rho^2 \cos\phi \, d\phi \, d\theta \, d\rho,$$

and

$$\frac{a-x}{\rho} = \cos\phi\cos\theta$$

Therefore equations (12) may be written

$$\begin{cases} X' = -k^2 \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_\epsilon^\rho \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho, \\ V' = \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_\epsilon^\rho \sigma\rho \cos\phi \, d\phi \, d\theta \, d\rho, \end{cases}$$

or

$$
(13)\quad \begin{cases} X' = -k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\rho} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho \\ \qquad\qquad + k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\epsilon} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho, \\ V' = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\rho} \sigma\rho \cos\phi \, d\phi \, d\theta \, d\rho - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\epsilon} \sigma\rho \cos\phi \, d\phi \, d\theta \, d\rho \end{cases}
$$

The first integrals in the right members are finite, determinate, and independent of ϵ Moreover, it follows from Art 72 that, for every positive value of ϵ,

$$
X' = k^2 \frac{\partial V'}{\partial a}
$$

Consider the second integrals of the right members of (13) Let

$$
X'' = k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\epsilon} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho,
$$

$$
V'' = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{\epsilon} \sigma\rho \cos\phi \, d\phi \, d\theta \, d\rho
$$

The differential elements of the first integral are positive or negative according as $\cos\theta$ is positive or negative, while all of those of the second integral are positive Let the first integral be divided so that the positive and negative elements occur separately, as

$$
\begin{aligned} X'' = k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\epsilon} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho \\ + k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{\epsilon} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho \\ + k^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\epsilon} \sigma \cos^2\phi \cos\theta \, d\phi \, d\theta \, d\rho \end{aligned}
$$

The differential elements are positive in the first two terms in the right member and negative in the third By hypothesis σ is finite

and continuous Let its maximum value in the sphere ϵ be σ_0 and its minimum σ_1 Then

$$k^2\sigma_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho$$

$$+k^2\sigma_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho$$

$$+k^2\sigma_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho$$

$$\leqq X''\leqq k^2\sigma_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho$$

$$+k^2\sigma_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{3\pi}{2}}^{2\pi}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho$$

$$+k^2\sigma_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_0^{\epsilon}\cos^2\phi\cos\theta\,d\phi\,d\theta\,d\rho,$$

$$0\geqq V''\geqq-\sigma_0\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\epsilon}\rho\cos\phi\,d\phi\,d\theta\,d\rho$$

Carrying out the integration, these inequalities become

$$\begin{cases} k^2\pi\epsilon(\sigma_1-\sigma_0)\leqq X''\leqq k^2\pi\epsilon(\sigma_0-\sigma_1), \\ \qquad 0\geqq V''\geqq-2\pi\sigma_0\epsilon^2 \end{cases}$$

Therefore

$$\begin{cases} \lim\limits_{\epsilon=0} X''=0, \\ \lim\limits_{\epsilon=0} V''=0 \end{cases}$$

Hence, at the limit,

$$\begin{cases} X=k^2\dfrac{\partial V}{\partial\alpha}, \text{ and similarly,} \\ Y=k^2\dfrac{\partial V}{\partial\beta}, \\ Z=k^2\dfrac{\partial V}{\partial\gamma}, \end{cases}$$

when the attracted particle is a part of the attracting mass, as well as when it is not

74 Level Surfaces The equation $V=c$, where c is a constant, defines what is called a *level surface*, or an *equipotential surface*, or, in

French, a *surface de niveau* In this equation the running coordinates are α, β, γ Any displacement $\delta\alpha$, $\delta\beta$, $\delta\gamma$, of the particle in this surface must fulfill the equation

$$\frac{\partial V}{\partial \alpha}\delta\alpha + \frac{\partial V}{\partial \beta}\delta\beta + \frac{\partial V}{\partial \gamma}\delta\gamma = 0,$$

or, as a consequence of (11),

$$X\delta\alpha + Y\delta\beta + Z\delta\gamma = 0 \qquad (14)$$

The direction cosines of the resultant attraction are proportional to X, Y, Z, and the direction cosines of the line of the displacement are proportional to $\delta\alpha$, $\delta\beta$, $\delta\gamma$ Since the sum of the products of these direction cosines in corresponding pairs is zero, it follows that the *resultant attraction is orthogonal to the level surfaces*

75 The Potential and Attraction of a Thin Homogeneous Circular Disc upon a Particle in its Axis Take the origin at the center of the disc whose radius is R Let the coordinates of P be α, 0, 0 Then

$$V = \int \frac{dm}{\rho} = \sigma \int_0^R \int_0^{2\pi} \frac{r\,dr\,d\theta}{\sqrt{\alpha^2 + r^2}}$$

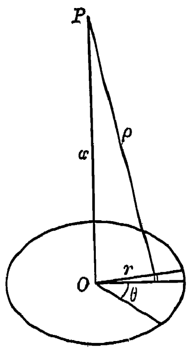

Fig 19

Integrating, it is found that

$$(15)\qquad \begin{cases} V = 2\pi\sigma\left[\sqrt{\alpha^2 + R^2} - \alpha\right], \\ X = k^2\dfrac{\partial V}{\partial \alpha} = 2\pi k^2\sigma\left[\dfrac{\alpha}{\sqrt{\alpha^2 + R^2}} - 1\right] \end{cases}$$

If α is kept constant and R is made to approach infinity as a limit, the attraction becomes

$$X = -2\pi k^2\sigma \qquad (16)$$

This equation does not depend upon a, therefore a thin circular disc of infinite extent attracts a particle above it with a force which is independent of its altitude Any number of superposed discs would act jointly in the same manner Hence, if the earth were a plane of infinite extent, as the ancients commonly supposed, bodies would gravitate toward it with constant forces at all altitudes, and the laws of falling bodies derived under the hypothesis of constant acceleration would be rigorously true

76 The Potential and Attraction of a Thin Homogeneous Spherical Shell upon an Interior or an Exterior Particle Let

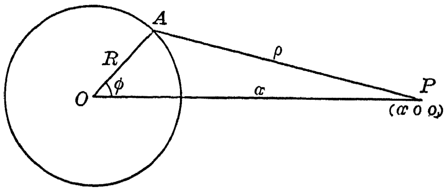

Fig 20

ϕ represent the angle between OP and the radius, and θ* the angle between the fundamental plane and the plane OAP Then

$$V = \int \frac{dm}{\rho} = \sigma \int_0^{\pi} \int_0^{2\pi} \frac{R^2 \sin\phi \, d\phi \, d\theta}{\rho} \tag{17}$$

One of the three variables ϕ, θ, ρ must be expressed in terms of the remaining two From the figure it is seen that

$$\rho^2 = a^2 + R^2 - 2aR\cos\phi ,$$

whence

$$\rho \, d\rho = aR \sin\phi \, d\phi \tag{18}$$

Then (17) becomes, if P is exterior,

$$V_E = \frac{R\sigma}{a} \int_{a-R}^{a+R} \int_0^{2\pi} d\rho \, d\theta , \tag{19}$$

and if P is interior,

$$V_I = \frac{R\sigma}{a} \int_{R-a}^{R+a} \int_0^{2\pi} d\rho \, d\theta \tag{20}$$

* It must be noticed that the ϕ and θ here are not the ordinary polar angles used elsewhere

The integrals of these equations are respectively

$$(21) \qquad \begin{cases} V_E = \dfrac{4\pi\sigma R^2}{a} = \dfrac{M}{a}, \\ V_I = 4\pi\sigma R = \dfrac{M}{R} \end{cases}$$

The x-components of attraction are respectively

$$(22) \qquad \begin{cases} X_E = k^2 \dfrac{\partial V_E}{\partial a} = -\dfrac{k^2 M}{a^2}, \\ X_I = k^2 \dfrac{\partial V_I}{\partial a} = 0, \end{cases}$$

which agree with the results obtained in Arts 66 and 70

The attraction of a solid homogeneous sphere may be also found at once Considering the shell as an element of the sphere, the M of (22) is given by the equation

$$M = 4\pi\sigma r^2 dr$$

Let $\overline{X}$ represent the attraction of the whole sphere $\overline{M}$, then

$$\overline{X} = -\frac{4k^2\pi\sigma}{a^2}\int_0^a r^2 dr = -\frac{4k^2}{3}\frac{\pi\sigma a^3}{a^2} = -k^2\frac{\overline{M}}{a^2}$$

77 Second Method of Computing the Attraction of a Homogeneous Sphere A very simple method will now be given of finding the attraction of a solid homogeneous sphere upon an exterior particle when it is known for interior particles It is a trivial matter in this case and is introduced only because of its simplicity The corresponding device in the case of the attractions of ellipsoids is of the greatest value, and constitutes Ivory's celebrated method

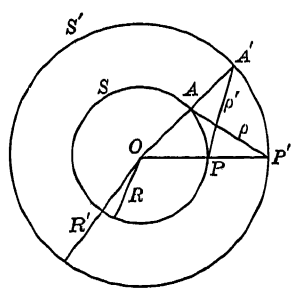

Fig 21

Let it be required to find the attraction of the sphere S upon the exterior particle P', supposing it is known how to find the attraction upon interior particles Construct the concentric sphere S' through

P' and suppose it has the same density as S. A one-to-one correspondence between the points on the surfaces of the two spheres is established by the relations

$$(23) \qquad \begin{cases} x = \dfrac{R}{R'} x', \\ y = \dfrac{R}{R'} y', \\ z = \dfrac{R}{R'} z' \end{cases}$$

The corresponding points are in a line passing through the common center of the spheres, and P corresponds to P'. Let X and X' represent the attractions of S' and S upon P and P' respectively. They are given by the equations

$$(24) \qquad \begin{cases} X = -k^2 \displaystyle\int_{(S')} \frac{R - x'}{\rho'^3}\, dm' = -k^2\sigma \iiint \frac{R - x'}{\rho'^3}\, dx'\, dy'\, dz', \\ X' = -k^2 \displaystyle\int_{(S)} \frac{R' - x}{\rho^3}\, dm = -k^2\sigma \iiint \frac{R' - x}{\rho^3}\, dx\, dy\, dz \end{cases}$$

But

$$(25) \qquad \begin{cases} -k^2\sigma \displaystyle\iiint \frac{R - x'}{\rho'^3}\, dx'\, dy'\, dz' = +k^2\sigma \iiint \frac{\partial\left(\frac{1}{\rho'}\right)}{\partial x}\, dx'\, dy'\, dz' \\ \qquad\qquad = k^2\sigma \displaystyle\iint \left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right) dy'\, dz', \\ -k^2\sigma \displaystyle\iiint \frac{R' - x}{\rho^3}\, dx\, dy\, dz = k^2\sigma \iint \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) dy\, dz, \end{cases}$$

where ρ_2 and ρ_1 are the extreme values of ρ obtained by integrating with respect to x. That is, the first integration gives the attraction of an elementary column extending through the sphere parallel to the x axis, and ρ_1 and ρ_2 are the distances from the attracted particle P' to the ends of this column. In completing the integration the sum of all of these elementary columns is taken. The corresponding statements with respect to the first equation of (25) are true.

Suppose these integrals are computed in such a manner that corresponding columns of the two spheres are always taken at the same time. Consider any two pairs of corresponding elements, as those at A and A'. For these $\rho = \rho'$, and this relation holds throughout the integration as arranged above. Hence it follows from equations (24) and (25) that

$$X' = k^2\sigma \iint \left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) dy\, dz = k^2\sigma \iint \left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right) dy\, dz$$

But, from (23), $$dy = \frac{R}{R'}\,dy', \quad dz = \frac{R}{R'}\,dz',$$

therefore $$X' = k^2\sigma\,\frac{R^2}{R'^2}\iint\left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right)dy'\,dz' = \frac{R^2}{R'^2}X$$

Let M represent the mass of the sphere S, and M' that of S' The attraction upon an interior particle is given by

$$X = -\frac{k^2M}{R^2},$$

therefore $$X' = -\frac{k^2M}{R'^2},$$

agreeing with results previously obtained (Arts 69, 70)

XI PROBLEMS

1 Prove by the limiting process that the potential and components of attraction have finite, determinate, values, and that equations (11) hold when the particle is on the surface of the attracting mass

2 Find the expression for the potential function for a particle exterior to the attracting body when the force varies inversely as the nth power of the distance

Ans $$V = \frac{1}{n-1}\int_{(M)}\frac{dm}{\rho^{n-1}}$$

3 Find by the limiting process for what values of n the potential in the last problem is finite and determinate when the particle is a part of the attracting mass

4 Show that the level surfaces for a straight homogeneous rod are prolate spheroids whose foci are the extremities of the rod.

5 Find the components of attraction of a uniform hemisphere whose radius is R upon a particle on its edge, (*a*) in the direction of the center of its base, (*b*) perpendicular to this in the plane of the base, (*c*) perpendicular to these two

Ans (*a*) $X = \frac{2}{3}\pi\sigma k^2R$, (*b*) $Y = 0$, (*c*) $Z = \frac{4}{3}\sigma k^2R$

6 Find the deviation of the plumb line due to a hemispherical hill of radius r and density σ_1 Let R represent the radius of the earth, assumed to be spherical, and σ_2 its mean density

Ans If λ is the angle of deviation,

$$\tan\lambda = \frac{\frac{2}{3}\pi\sigma_1 r}{\frac{4}{3}\pi\sigma_2 R - \frac{4}{3}\sigma_1 r} = \frac{\frac{1}{2}\pi\sigma_1 r}{\pi\sigma_2 R - \sigma_1 r},$$

or $$\tan\lambda = \frac{1}{2}\frac{\sigma_1}{\sigma_2}\frac{r}{R} \text{ approximately}$$

7 Prove that if the attraction varies directly as the distance, a body of any shape attracts a particle as though its whole mass were concentrated at its center of mass

78 The Potential and Attraction of a Solid Homogeneous Oblate Spheroid upon a Distant Particle The planets are very nearly oblate spheroids, and they are nearly enough homogeneous so that the results obtained in this article will represent the actual facts with sufficient approximation

Suppose the attracted particle is remote compared to the dimensions of the attracting spheroid Take the origin of coordinates at the center

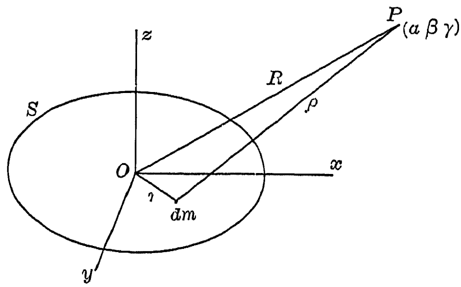

Fig 22

of the spheroid with the z-axis coinciding with the axis of revolution Let R represent the distance from O to P, and r the distance from O to the element of mass Then

$$(26)\qquad \begin{cases} V = \int_{(S)} \dfrac{dm}{\rho}, \\ \rho = \sqrt{(\alpha - x)^2 + (\beta - y)^2 + (\gamma - z)^2}, \\ R = \sqrt{\alpha^2 + \beta^2 + \gamma^2}, \\ r = \sqrt{x^2 + y^2 + z^2} \end{cases}$$

It follows from these equations that

$$\frac{1}{\rho} = \frac{1}{\sqrt{R^2 + r^2 - 2(\alpha x + \beta y + \gamma z)}} = \frac{1}{R\sqrt{1 + \dfrac{r^2 - 2(\alpha x + \beta y + \gamma z)}{R^2}}}$$

Let $\frac{x}{R}$, $\frac{y}{R}$, and $\frac{z}{R}$ be taken as small quantities of the first order, then expanding by the binomial theorem, it is found that, up to small quantities of the third order,

$$\frac{1}{\rho} = \frac{1}{R}\left\{1 + \frac{\alpha x + \beta y + \gamma z}{R^2} - \frac{r^2}{2R^2} \right.$$
$$\left. + \frac{3}{2}\frac{(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 + 2\alpha\beta xy + 2\beta\gamma yz + 2\gamma\alpha zx)}{R^4} + \cdots \right\}$$

Therefore

$$(27)\quad \begin{cases} V = \dfrac{1}{R}\displaystyle\int dm + \dfrac{\alpha}{R^3}\int x\,dm + \dfrac{\beta}{R^3}\int y\,dm + \dfrac{\gamma}{R^3}\int z\,dm - \dfrac{1}{2R^3}\int r^2\,dm \\ \quad + \dfrac{3}{2}\dfrac{\alpha^2}{R^5}\displaystyle\int x^2\,dm + \dfrac{3}{2}\dfrac{\beta^2}{R^5}\int y^2\,dm + \dfrac{3}{2}\dfrac{\gamma^2}{R^5}\int z^2\,dm \\ \quad + \dfrac{3\alpha\beta}{R^5}\displaystyle\int xy\,dm + \dfrac{3\beta\gamma}{R^5}\int yz\,dm + \dfrac{3\gamma\alpha}{R^5}\int zx\,dm + \end{cases}$$

Let M represent the mass of the spheroid, then

$$\int dm = M,$$

and, since the origin is at the center of gravity,

$$\int x\,dm = 0, \quad \int y\,dm = 0, \quad \int z\,dm = 0$$

Let σ represent the density, then

$$\begin{cases} dm = \sigma r^2 \cos\phi\, d\phi\, d\theta\, dr, \\ x = r\cos\phi\cos\theta, \\ y = r\cos\phi\sin\theta, \\ z = r\sin\phi, \end{cases}$$

and (27) becomes

$$\begin{aligned} V = \frac{M}{R} &- \frac{\sigma}{2R^3}\iiint r^4 \cos\phi\, d\phi\, d\theta\, dr + \frac{3}{2}\frac{\alpha^2\sigma}{R^5}\iiint r^4\cos^3\phi\cos^2\theta\, d\phi\, d\theta\, dr \\ &+ \frac{3}{2}\frac{\beta^2\sigma}{R^5}\iiint r^4\cos^3\phi\sin^2\theta\, d\phi\, d\theta\, dr + \frac{3}{2}\frac{\gamma^2\sigma}{R^5}\iiint r^4\sin^2\phi\cos\phi\, d\phi\, d\theta\, dr \\ &+ \frac{3\alpha\beta\sigma}{R^5}\iiint r^4\cos^3\phi\sin\theta\cos\theta\, d\phi\, d\theta\, dr \\ &+ \frac{3\beta\gamma\sigma}{R^5}\iiint r^4\sin\phi\cos^2\phi\sin\theta\, d\phi\, d\theta\, dr \\ &+ \frac{3\gamma\alpha\sigma}{R^5}\iiint r^4\sin\phi\cos^2\phi\cos\theta\, d\phi\, d\theta\, dr + \quad , \end{aligned}$$

where the limits of integration are for r, 0 and r, for ϕ, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and for θ, 0 and 2π Since r and ϕ are independent of θ, the integration may be performed with respect to θ first, giving

$$
(28)\quad \left\{
\begin{aligned}
V = {} & \frac{M}{R} - \frac{\pi\sigma}{R^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^r r^4 \cos\phi \, d\phi \, dr \\
& + \frac{3}{2}\frac{\pi\alpha^2\sigma}{R^5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^r r^4 \cos^3\phi \, d\phi \, dr \\
& + \frac{3}{2}\frac{\pi\beta^2\sigma}{R^5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^r r^4 \cos^3\phi \, d\phi \, dr \\
& + \frac{3\pi\gamma^2\sigma}{R^5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^r r^4 \sin\phi \cos\phi \, d\phi \, dr + \cdots,
\end{aligned}
\right.
$$

the last three integrals being zero

The next integration must be made with respect to r, as this variable depends upon ϕ Let the major and minor semi-axes of a meridian section be a and b respectively, and let e be the eccentricity Then

$$
r^2 = \frac{b^2}{1 - e^2 \cos^2\phi}
$$

Integrating (28) with respect to r and expanding in powers of e, it is found that, up to terms of the second order inclusive,

$$
\begin{aligned}
V = {} & \frac{M}{R} - \frac{\pi\sigma b^5}{5R^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1 + \tfrac{5}{2}e^2\cos^2\phi + \cdots\right)\cos\phi \, d\phi \\
& + \frac{3}{10}\frac{\pi\sigma b^5}{R^5}(\alpha^2 + \beta^2)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1 + \tfrac{5}{2}e^2\cos^2\phi + \cdots\right)\cos^3\phi \, d\phi \\
& + \frac{3}{5}\frac{\pi\sigma b^5\gamma^2}{R^5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1 + \tfrac{5}{2}e^2\cos^2\phi + \cdots\right)\sin^2\phi\cos\phi \, d\phi \\
& + \cdots
\end{aligned}
$$

Integrating with respect to ϕ and arranging in powers of e, it is found that

$$
V = \frac{M}{R} + \frac{2}{15}\frac{\pi\sigma b^5}{R^5}(\alpha^2 + \beta^2 - 2\gamma^2)e^2 + \cdots
$$

But

$$
M = \tfrac{4}{3}\pi\sigma a^2 b,
$$

$$
b^2 = a^2(1 - e^2),
$$

therefore

$$V = \frac{M}{R}\left[1 + \frac{b^2}{10}\frac{(\alpha^2 + \beta^2 - 2\gamma^2)}{R^4} e^2 + \quad\right] \tag{29}$$

The components of attraction are

$$\begin{cases} X = k^2 \dfrac{\partial V}{\partial \alpha} = -\dfrac{k^2 M \alpha}{R^3}\left[1 + \dfrac{3}{10} b^2 \dfrac{(\alpha^2 + \beta^2 - 4\gamma^2)}{R^4} e^2 + \quad\right], \\ Y = k^2 \dfrac{\partial V}{\partial \beta} = -\dfrac{k^2 M \beta}{R^3}\left[1 + \dfrac{3}{10} b^2 \dfrac{(\alpha^2 + \beta^2 - 4\gamma^2)}{R^4} e^2 + \quad\right], \\ Z = k^2 \dfrac{\partial V}{\partial \gamma} = -\dfrac{k^2 M \gamma}{R^3}\left[1 + \dfrac{3}{10} b^2 \dfrac{3(\alpha^2 + \beta^2) - 2\gamma^2}{R^4} e^2 + \quad\right] \end{cases} \tag{30}$$

If the spheroid should become a sphere of the same mass, the expressions for the components of attraction would reduce to the first terms of the right members of equations (30) If the attracted particle is in the plane of the equator of the attracting spheroid, then $\gamma = 0$, and if it is in the polar line, then $\alpha = \beta = 0$ Hence it follows from (30) that *the attraction of an oblate spheroid upon a particle in the plane of its equator is greater than that of a sphere of equal mass, and in the polar line, less than that of a sphere of equal mass* As the particle recedes from the attracting body the attraction approaches that of a sphere of equal mass Therefore, *as the particle recedes in the plane of the equator the attraction decreases more rapidly than the square of the distance increases, and as it approaches, the attraction increases more rapidly than the square of the distance decreases* The opposite results are true when the particle is in the polar line

79 The Potential and Attraction of a Solid Homogeneous Ellipsoid upon a Unit Particle in its Interior Let the equation of the surface of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0, \tag{31}$$

and let the attracted particle be situated at the interior point (α, β, γ) Take this point for the origin of the polar coordinates ρ, θ, and ϕ Taking the fundamental planes of this system parallel to those of the first system, these variables are related to the rectangular coordinates by the equations

$$\begin{cases} x = \alpha + \rho\cos\phi\cos\theta, \\ y = \beta + \rho\cos\phi\sin\theta, \\ z = \gamma + \rho\sin\phi \end{cases} \tag{32}$$

The potential of the ellipsoid upon the unit particle P is

$$V=\int_{(M)}\frac{dm}{\rho}=\sigma\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{\rho_1}\rho\cos\phi\, d\phi\, d\theta\, d\rho$$

Since the value of ρ depends upon the polar angles the integration must be made first with respect to this variable The integration gives

$$(33)\qquad V=\frac{\sigma}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\rho_1\cos\phi\, d\phi\, d\theta$$

To express ρ_1 in terms of the polar angles substitute (32) in (31), whence

$$(34)\qquad A\rho_1^2+2B\rho_1+C=0,$$

where

$$(35)\qquad \begin{cases} A=\dfrac{\cos^2\phi\cos\theta}{a^2}+\dfrac{\cos^2\phi\sin\theta}{b^2}+\dfrac{\sin\phi}{c^2}, \\ B=\dfrac{\alpha\cos\phi\cos\theta}{a^2}+\dfrac{\beta\cos\phi\sin\theta}{b}+\dfrac{\gamma\sin\phi}{c^2}, \\ C=\dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b^2}+\dfrac{\gamma}{c^2}-1 \end{cases}$$

From (34) it is found that

$$\rho_1=\frac{-B\pm\sqrt{B-AC}}{A}$$

The only ρ_1 having a meaning in this problem is positive A is essentially positive, and C is negative because (α, β, γ) is within the ellipsoidal surface, therefore the positive sign must be taken before the radical Substituting this value of ρ_1 in (33), it is found that

$$(36)\qquad V=\frac{\sigma}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{(2B-2B\sqrt{B^2-AC}-AC)}{A}\cos\phi\, d\phi\, d\theta$$

Consider the integral

$$V_1=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{B\sqrt{B^2-AC}}{A}\cos\phi\, d\phi\, d\theta$$

It follows from the expression for B that the differential elements corresponding to $\theta=\theta_0$, $\phi=\phi_0$ and to $\theta=\pi+\theta_0$, $\phi=-\phi_0$ are equal in numerical value but opposite in sign Since all the elements entering in the integral may be paired in this way, it follows that $V_1=0$, after which (36) becomes

$$(37)\quad \begin{cases} V = \dfrac{\sigma}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left\{\frac{\cos^2\phi\cos^2\theta}{a^2}\left(\frac{2\alpha^2}{a^2}-C\right)+\frac{\cos^2\phi\sin^2\theta}{b^2}\left(\frac{2\beta^2}{b^2}-C\right)\right. \\ \qquad\qquad\qquad\qquad \left.+\dfrac{\sin^2\phi}{c^2}\left(\dfrac{2\gamma^2}{c^2}-C\right)\right\}\dfrac{\cos\phi\, d\phi\, d\theta}{A^2} \\ \quad + 2\sigma\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\left\{\frac{\alpha\beta\cos^2\phi\sin\theta\cos\theta}{a^2b^2}+\frac{\beta\gamma\sin\phi\cos\phi\sin\theta}{b^2c^2}\right. \\ \qquad\qquad\qquad\qquad \left.+\dfrac{\gamma\alpha\sin\phi\cos\phi\cos\theta}{c^2a^2}\right\}\dfrac{\cos\phi\, d\phi\, d\theta}{A^2} \end{cases}$$

By comparing the elements properly paired, it is seen that the second integral is zero

Let

$$(38)\qquad W = \frac{\sigma}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{\cos\phi\, d\phi\, d\theta}{\dfrac{\cos^2\phi\cos^2\theta}{a^2}+\dfrac{\cos^2\phi\sin^2\theta}{b^2}+\dfrac{\sin^2\phi}{c^2}},$$

then (37) may be written

$$(39)\qquad V = -CW + \frac{\alpha^2}{a}\frac{\partial W}{\partial a}+\frac{\beta^2}{b}\frac{\partial W}{\partial b}+\frac{\gamma^2}{c}\frac{\partial W}{\partial c}$$

Since W is independent of α, β, and γ, the components of attraction are

$$(40)\qquad \begin{cases} X = \dfrac{2\alpha}{a}k^2\dfrac{\partial W}{\partial a} - k^2 W\dfrac{\partial C}{\partial \alpha}, \\ Y = \dfrac{2\beta}{b}k^2\dfrac{\partial W}{\partial b} - k^2 W\dfrac{\partial C}{\partial \beta}, \\ Z = \dfrac{2\gamma}{c}k^2\dfrac{\partial W}{\partial c} - k^2 W\dfrac{\partial C}{\partial \gamma} \end{cases}$$

For a given ellipsoid W is a constant, and the equation of the level surfaces has the form

$$C_1\alpha^2 + C_2\beta^2 + C_3\gamma^2 = \text{constant},$$

which is the equation of concentric similar ellipsoids, whose axes are proportional to $C_1^{-\frac{1}{2}}$, $C_2^{-\frac{1}{2}}$, and $C_3^{-\frac{1}{2}}$

Let

$$\begin{cases} M = \dfrac{\cos^2\phi}{a^2}+\dfrac{\sin^2\phi}{c^2}, \\ N = \dfrac{\cos^2\phi}{b^2}+\dfrac{\sin^2\phi}{c^2}, \end{cases}$$

then (38) becomes

$$W = \frac{\sigma}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{\cos\phi\, d\phi\, d\theta}{M\cos^2\theta+N\sin^2\theta} = 4\sigma\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\cos\phi\, d\phi\, d\theta}{M\cos^2\theta+N\sin^2\theta}$$

M and N are independent of θ, hence, integrating with respect to this variable, it is found that*

$$(41)\qquad W = 2\pi\sigma \int_0^{\frac{\pi}{2}} \frac{\cos\phi\, d\phi}{\sqrt{MN}}$$

$$= 2\pi\sigma\, abc^2 \int_0^{\frac{\pi}{2}} \frac{\cos\phi\, d\phi}{\sqrt{(a^2\sin^2\phi + c^2\cos^2\phi)(b^2\sin^2\phi + c^2\cos^2\phi)}}$$

To return to the symmetry in a, b, and c which existed in (38), Rodriguez introduced the transformation

$$\sin\phi = \frac{c}{\sqrt{c^2 + s}},$$

whence

$$W = \pi\sigma\, abc \int_0^{\infty} \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}$$

Forming the derivatives with respect to a, b, and c, and substituting in (39), it follows that

$$(42)\qquad V = \pi\sigma abc \int_0^{\infty} \left(1 - \frac{\alpha^2}{a^2+s} - \frac{\beta^2}{b^2+s} - \frac{\gamma^2}{c^2+s}\right) \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}$$

The components of attraction are

$$(43)\qquad \begin{cases} X = k^2 \dfrac{\partial V}{\partial \alpha} = -2\pi\sigma\, abc\,\alpha k^2 \displaystyle\int_0^{\infty} \frac{ds}{(a^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \\[2ex] Y = k^2 \dfrac{\partial V}{\partial \beta} = -2\pi\sigma\, abc\,\beta k^2 \displaystyle\int_0^{\infty} \frac{ds}{(b^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \\[2ex] Z = k^2 \dfrac{\partial V}{\partial \gamma} = -2\pi\sigma\, abc\,\gamma k^2 \displaystyle\int_0^{\infty} \frac{ds}{(c^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}} \end{cases}$$

Equation (41) is homogeneous of the second degree in a, b, and c, and therefore $\frac{\partial V}{\partial \alpha}$, $\frac{\partial V}{\partial \beta}$, $\frac{\partial V}{\partial \gamma}$, computed from (39), are homogeneous of degree zero in the same quantities. It follows, therefore, that if a, b, and c are increased by any factor ν the components of attraction X, Y, and Z, will not be changed, or, *the elliptic homoeoid contained between the ellipsoidal surfaces whose axes are a, b, c, and νa, νb, and νc attracts the interior particle P equally in opposite directions* (Compare Art. 67.)

The component of attraction, X, is independent of β and γ and involves α to the first degree, therefore *the x-component of attraction is proportional to the x-coordinate of the particle and is constant everywhere within the ellipsoid in the plane $x = \alpha$.* Similar results are true for the two other coordinates.

* Letting $\tan\theta = x$, the integral reduces to one of the standard forms.

Suppose the notation has been chosen so that $a > b > c$. Then (41) may be put in a very convenient form by the substitution

$$\begin{cases} \sin\phi = u, \\ \quad a^2 = c^2(1+\lambda^2), \\ \quad b^2 = c^2(1+\mu^2), \end{cases}$$

which gives

$$(44) \qquad W = 2\pi\sigma ab \int_0^1 \frac{du}{\sqrt{(1+\lambda^2 u^2)(1+\mu^2 u^2)}}$$

This is an elliptic integral of the first kind, and may be readily computed when the body is nearly spherical by expanding in powers of λ^2 and μ^2, and integrating term by term

XII PROBLEMS

1 Discuss the level surfaces given by equation (29)

2 Set up the expressions for the components of attraction instead of that for the potential as in Art. 79 Determine what parts of the integrals vanish, integrate with respect to θ, and show that the results are

$$\begin{cases} X = -4\pi\sigma bc\alpha k^2 \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\cos\phi\, d\phi}{\sqrt{(b^2\sin^2\phi + a^2\cos^2\phi)(c^2\sin^2\phi + a^2\cos^2\phi)}}, \\ Y = -4\pi\sigma ca\beta k^2 \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\cos\phi\, d\phi}{\sqrt{(c^2\sin^2\phi + b^2\cos^2\phi)(a^2\sin^2\phi + b^2\cos^2\phi)}}, \\ Z = -4\pi\sigma ab\gamma k^2 \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2\phi\cos\phi\, d\phi}{\sqrt{(a^2\sin^2\phi + c^2\cos^2\phi)(b^2\sin^2\phi + c^2\cos^2\phi)}} \end{cases}$$

Hint Derive the results for Z, and since it is immaterial in what order the axes are chosen, derive the others by a permutation of the letters a, b, c

3 Transform the equations of problem 2 by

$$\sin\phi = \frac{a}{\sqrt{a^2+s}}, \quad \sin\phi = \frac{b}{\sqrt{b^2+s}}, \quad \sin\phi = \frac{c}{\sqrt{c^2+s}}$$

respectively, and show that equations (43) result

4 Show that the potential of an ellipsoid upon a particle at its center is

$$V_0 = \pi\sigma abc \int_0^\infty \frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}} = W$$

5 From the value of V_0 and equations (43) derive the value of the potential (42)

6 Transform the equations of problem 2 so that they take the form

$$\int \frac{u^2 du}{\sqrt{(1 \pm \lambda^2 u^2)(1 \pm \mu^2 u^2)}}$$

80 The Attraction of a Solid Homogeneous Ellipsoid upon an Exterior Particle Ivory's Method The integrals become so complicated in the case of an exterior particle that the components of attraction never have been found by direct integration They are computed indirectly by expressing them in terms of the components of attraction of a related ellipsoid upon particles in its interior This artifice constitutes Ivory's method*

Let it be required to find the attraction of the ellipsoid E upon the exterior particle, P', at $(\alpha', \beta', \gamma')$ Let the semi-axes of E be a, b, and c Construct through P' an ellipsoid E', confocal with E, with the

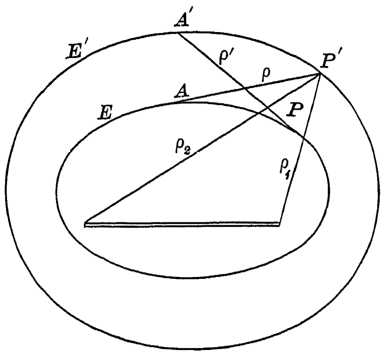

Fig 23

semi-axes a', b', c', and suppose it has the same density as E The axes of the two ellipsoids are related by the equations

$$(45) \qquad \begin{cases} a' = \sqrt{a^2 + \kappa}, \\ b' = \sqrt{b^2 + \kappa}, \\ c' = \sqrt{c^2 + \kappa}, \end{cases}$$

where κ is defined by the equation

$$(46) \qquad \frac{\alpha'^2}{a^2+\kappa} + \frac{\beta'^2}{b^2+\kappa} + \frac{\gamma'^2}{c^2+\kappa} - 1 = 0$$

The only value of κ having a meaning in this problem is real and positive Equation (46) is a cubic in κ and has one positive and two negative roots, for, the left member considered as a function of κ is negative when $\kappa = +\infty$, positive, when $\kappa = 0$ (because $(\alpha', \beta', \gamma')$ is exterior to the ellipsoid E), positive, when $\kappa = -c^2 + \epsilon$ (where ϵ is a very small quantity), negative, when $\kappa = -c^2 - \epsilon$, positive, when

* *Philosophical Transactions*, 1809

$\kappa = -b^2 + \epsilon$, negative, when $\kappa = -b^2 - \epsilon$, positive, when $\kappa = -a^2 + \epsilon$, negative, when $\kappa = -a^2 - \epsilon$, and negative when $\kappa = -\infty$ The graph of the function is given in Fig 24

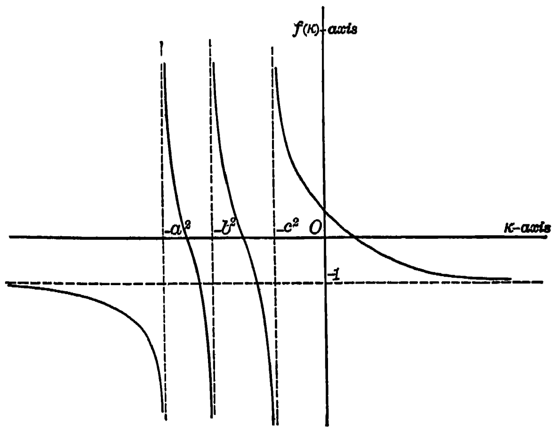

Fig 24

Taking the positive root of (46), a', b', and c' are determined uniquely

A one-to-one correspondence between the points upon the two ellipsoids will now be established by the equations (compare Art 77)

$$(47) \qquad x' = \frac{a'}{a}x, \quad y' = \frac{b'}{b}y, \quad z' = \frac{c'}{c}z$$

Let P be the point corresponding to P' It will now be shown that the attraction of E upon P' is related in a very simple manner to the attraction of E' upon P

Let X, Y, and Z represent the components of attraction of E' upon the interior particle P, (α, β, γ) They may be computed by the methods of Art 79, and will be supposed known Let X', Y', and Z' be the components of attraction of E upon P', which are required. The expressions for the x-components are

$$(48) \quad \begin{cases} X = -k^2\sigma \displaystyle\iiint \frac{\alpha - x'}{\rho'^3}\, dx'\, dy'\, dz' = k^2\sigma \iiint \frac{\partial\left(\frac{1}{\rho'}\right)}{\partial x'}\, dx'\, dy'\, dz', \\[2ex] X' = -k^2\sigma \displaystyle\iiint \frac{\alpha' - x}{\rho^3}\, dx\, dy\, dz = k^2\sigma \iiint \frac{\partial\left(\frac{1}{\rho}\right)}{\partial x}\, dx\, dy\, dz \end{cases}$$

Performing the integration with respect to x, it follows that

$$
(49) \qquad \begin{cases} X = k^2\sigma \displaystyle\iint \left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right) dy'\, dz', \\[2ex] X' = k^2\sigma \displaystyle\iint \left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) dy\, dz \end{cases}
$$

The ρ and ρ_1 are the distances from P' to the ends of the elementary column obtained by integrating with respect to x The solution is completed by integrating over the whole surface of E The first equation of (49) is similar

Now X' will be related to X in a simple manner by the aid of the following lemma

If P and A are any two points on the surface of E, and if P' and A' are the respective corresponding points on the surface of E', then the distances $\overline{PA'}$ and $\overline{P'A}$ are equal

Let $\overline{PA'} = \rho'$, and $\overline{AP'} = \rho$ Then $\rho = \rho'$ For, let the coordinates of P and A be respectively x_1, y_1, z_1, and x_2, y_2, z_2, and of P' and A', x_1', y_1', z_1', and x_2', y_2', z_2' Then

$$
\begin{cases} \rho'^2 = (x_1 - x_2')^2 + (y_1 - y_2')^2 + (z_1 - z_2')^2, \\ \rho^2 = (x_2 - x_1')^2 + (y_2 - y_1')^2 + (z_2 - z_1')^2 \end{cases}
$$

Making use of equations (45) and (47), it is found that

$$
\rho'^2 - \rho^2 = \kappa \left(\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2}\right) - \kappa \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2}\right)
$$

Since P and A are on the surface of the ellipsoid whose semi-axes are a, b, and c, each parenthesis equals unity Therefore $\rho'^2 - \rho^2 = 0$, or $\rho = \rho'$ Q E D

Suppose the integrals (49) are computed so that the elements at *corresponding points* of the two surfaces are always taken simultaneously Then $\rho_1 = \rho_1'$ and $\rho_2 = \rho'$ throughout the integration Moreover, from (47), $dy = \dfrac{b}{b'}\, dy'$, $dz = \dfrac{c}{c'}\, dz$ Therefore

$$
(50) \qquad \begin{cases} X = k^2\sigma \displaystyle\iint \left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right) dy'\, dz', \\[2ex] X' = k^2\sigma \dfrac{bc}{b'c'} \displaystyle\iint \left(\frac{1}{\rho_2'} - \frac{1}{\rho_1'}\right) dy'\, dz' = \frac{bc}{b'c'} X, \end{cases}
$$

and similarly

$$
(51) \qquad \begin{cases} Y' = \dfrac{ca}{c'a'}\, Y, \\[2ex] Z' = \dfrac{ab}{a'b'}\, Z \end{cases}
$$

The letters a, b, c and s of equations (43) should be given accents to agree with the notations of this article, and, since P and P' are corresponding points, $\alpha=\frac{a}{a'}\alpha'$, $\beta=\frac{b}{b'}\beta'$, $\gamma=\frac{c}{c'}\gamma'$ Making these changes in equations (43) and substituting them in (50) and (51), it is found that

$$\left\{\begin{aligned} X' &= -2\pi\sigma abck^2\alpha'\int_0^\infty \frac{ds'}{(a'^2+s')\sqrt{(a'^2+s')(b'^2+s')(c'^2+s')}}, \\ Y' &= -2\pi\sigma abck^2\beta'\int_0^\infty \frac{ds'}{(b'^2+s')\sqrt{(a'^2+s)(b'^2+s')(c'^2+s')}}, \\ Z' &= -2\pi\sigma abck^2\gamma'\int_0^\infty \frac{ds'}{(c'^2+s')\sqrt{(a'^2+s')(b^2+s')(c'^2+s')}} \end{aligned}\right.$$

By equations (45)

$$a'^2=a^2+\kappa,\quad b'^2=b^2+\kappa,\quad c'^2=c^2+\kappa,$$

hence, letting $s=s'+\kappa$, it follows that

$$(52)\qquad \left\{\begin{aligned} X' &= -2\pi\sigma abck^2\alpha'\int_\kappa^\infty \frac{ds}{(a^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \\ Y' &= -2\pi\sigma abck^2\beta'\int_\kappa^\infty \frac{ds}{(b^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}}, \\ Z' &= -2\pi\sigma abck^2\gamma'\int_\kappa^\infty \frac{ds}{(c^2+s)\sqrt{(a^2+s)(b^2+s)(c^2+s)}} \end{aligned}\right.$$

It follows from equations (40) and (41) that the components of attraction for interior particles are homogeneous of degree zero in a, b, and c, and that they are proportional to the respective coordinates of the attracted particle Let X, as above, represent the attraction of the ellipsoid E', whose semi-axes are a', b', c', upon the interior particle at (α, β, γ), let X'' represent the attraction of E' upon an interior particle at $(\alpha'', \beta'', \gamma'')$, which will be supposed to be related to (α, β, γ) by equations of the same form as (47) Then it follows that

$$\frac{X''}{X}=\frac{\alpha''}{\alpha},\quad \frac{Y''}{Y}=\frac{\beta''}{\beta},\quad \frac{Z''}{Z}=\frac{\gamma''}{\gamma}$$

Let the point $(\alpha'', \beta'', \gamma'')$, always remaining corresponding to (α, β, γ), approach the surface of E' as a limit Then at the limit

$$\frac{X''}{X}=\frac{a'}{a},\qquad \frac{Y''}{Y}=\frac{b'}{b},\qquad \frac{Z''}{Z}=\frac{c'}{c}$$

Combining these equations with (50) and (51), it follows that

$$(53)\qquad \frac{X''}{X'}=\frac{Y''}{Y'}=\frac{Z''}{Z'}=\frac{a'b'c'}{abc}=\frac{M'}{M}$$

That is, *the attraction of a solid ellipsoid upon an exterior particle is to the attraction of a confocal ellipsoid passing through the particle, as the mass of the first ellipsoid is to that of the second ellipsoid*

Consider another ellipsoid confocal with the one passing through the particle and interior to it, by the same reasoning the ratios of the components of attraction of these two ellipsoids are as their masses Let X''', Y''', Z'' be the components of attraction of the new ellipsoid whose semi-axes are a''', b''', c''' Then

$$\frac{X''}{X'''}=\frac{Y''}{Y'''}=\frac{Z''}{Z'''}=\frac{a'b'c'}{a''b''c'''}=\frac{M'}{M'''}$$

Combining this proportion with (53), it is found that

$$\frac{X'}{X'''}=\frac{Y'}{Y'''}=\frac{Z'}{Z'''}=\frac{M}{M'''}$$

Therefore, *two confocal ellipsoids attract particles which are exterior to both of them in the same direction and with forces which are proportional to their masses* This theorem was found by Maclaurin and Lagrange for ellipsoids of revolution, and was extended by Laplace to the general case where the three axes are unequal It is established most easily, however, by Ivory's method as above, and it is frequently called Ivory's theorem

Equations (52) may be transformed into the normal form of elliptic integrals of the first kind by putting, in the first, $\frac{a}{\sqrt{a^2+s}}=u$, in the second, $\frac{b}{\sqrt{b^2+s}}=u$, and in the third, $\frac{c}{\sqrt{c^2+s}}=u$ The results of the substitutions are

$$(54)\quad\begin{cases} X'=-4\pi\sigma bck^2\alpha'\displaystyle\int_0^{\frac{a}{\sqrt{a^2+\kappa}}}\frac{u^2du}{\sqrt{[a^2-(a^2-b^2)u^2][a^2-(a^2-c^2)u^2]}}, \\ Y=-4\pi\sigma cak^2\beta'\displaystyle\int_0^{\frac{b}{\sqrt{b^2+\kappa}}}\frac{u^2du}{\sqrt{[b^2-(b^2-c^2)u^2][b^2-(b^2-a^2)u]}}, \\ Z'=-4\pi\sigma abk^2\gamma'\displaystyle\int_0^{\frac{c}{\sqrt{c^2+\kappa}}}\frac{u^2du}{\sqrt{[c^2-(c^2-a^2)u^2][c^2-(c^2-b^2)u^2]}} \end{cases}$$

When the attracted particle is in the interior of the ellipsoid the forms of the integrals are the same except that the upper limits are unity

81 The Attraction of Spheroids The components of attraction will be obtained from (54), which hold for exterior particles

Suppose the figure is an oblate spheroid and that $a=b>c$, let e represent the eccentricity of a meridian section Then

$$c^2 = a^2(1-e^2),$$

and equations (54) become

$$(55)\quad \begin{cases} \dfrac{X'}{\alpha'} = \dfrac{Y'}{\beta'} = -4\pi\sigma k^2\sqrt{1-e^2}\displaystyle\int_0^{\frac{a}{\sqrt{a^2+\kappa}}} \frac{u^2du}{\sqrt{1-e^2u^2}}, \\ \dfrac{Z'}{\gamma'} = -4\pi\sigma k^2 \displaystyle\int_0^{\frac{c}{\sqrt{c^2+\kappa}}} \frac{u^2du}{1-e^2+e^2u^2} \end{cases}$$

The integrals of these equations are

$$(56)\quad \begin{cases} \dfrac{X'}{\alpha'} = \dfrac{Y'}{\beta'} = -2\pi\sigma k^2\dfrac{\sqrt{1-e^2}}{e^3}\left[\dfrac{-ae}{\sqrt{a^2+\kappa}}\sqrt{1-\dfrac{a^2e^2}{a^2+\kappa}} + \sin^{-1}\left(\dfrac{ae}{\sqrt{a^2+\kappa}}\right)\right], \\ \dfrac{Z'}{\gamma'} = -4\pi\sigma\dfrac{k^2}{e^3}\left[\dfrac{ce}{\sqrt{c^2+\kappa}} - \sqrt{1-e^2}\tan^{-1}\left(\dfrac{ce}{\sqrt{(1-e^2)(c^2+\kappa)}}\right)\right] \end{cases}$$

The components of attraction for interior particles are obtained from these equations by putting $\kappa = 0$

Suppose the figure is a prolate spheroid, and that $a=b<c$ Then $a^2=b^2=c^2(1-\epsilon^2)$, and equations (54) become

$$(57)\quad \begin{cases} \dfrac{X'}{\alpha'} = \dfrac{Y'}{\beta'} = -4\pi\sigma k^2 \displaystyle\int_0^{\frac{a}{\sqrt{a^2+\kappa}}} \frac{u^2du}{\sqrt{1-\epsilon^2+\epsilon^2u^2}}, \\ \dfrac{Z'}{\gamma'} = -4\pi\sigma k^2(1-\epsilon^2)\displaystyle\int_0^{\frac{c}{\sqrt{c^2+\kappa}}} \frac{u^2du}{1-\epsilon^2u^2} \end{cases}$$

The integrals of these equations are

$$(58)\quad \begin{cases} \dfrac{X'}{\alpha'} = \dfrac{Y'}{\beta'} = -\dfrac{2\pi\sigma k^2}{\epsilon^3}\left[\dfrac{a\epsilon}{\sqrt{a^2+\kappa}}\sqrt{1-\epsilon^2+\dfrac{a^2\epsilon^2}{a^2+\kappa}} \right. \\ \qquad \left. -(1-\epsilon^2)\log\left(\dfrac{a\epsilon}{\sqrt{(1-\epsilon^2)(a^2+\kappa)}} + \sqrt{1+\dfrac{a^2\epsilon^2}{(1-\epsilon^2)(a^2+\kappa)}}\right)\right], \\ \dfrac{Z'}{\gamma'} = -2\pi\sigma k^2\dfrac{(1-\epsilon^2)}{\epsilon^3}\left[\log\left(\dfrac{1+\dfrac{c\epsilon}{\sqrt{c^2+\kappa}}}{1-\dfrac{c\epsilon}{\sqrt{c^2+\kappa}}}\right)\right] \end{cases}$$

When the particle is interior to the spheroid the equations for the components of attraction are the same except that $\kappa = 0$

82 The Attraction at the Surfaces of Spheroids The components of attraction for an interior particle, which are obtained in the case of an oblate spheroid from (56) by putting $\kappa = 0$, are, omitting the accents,

$$(59)\quad \begin{cases} \dfrac{X}{\alpha} = \dfrac{Y}{\beta} = -\,2\pi\sigma k\ \dfrac{\sqrt{1-e}}{e^3}\left[-e\sqrt{1-e^2} + \sin^{-1} e\right], \\[2ex] \dfrac{Z}{\gamma} = -\,4\pi\sigma\dfrac{k}{e^3}\left[e - \sqrt{1-e^2}\tan^{-1}\left(\dfrac{e}{\sqrt{1-e}}\right)\right] \end{cases}$$

The limits of these expressions as the attracted particle approaches the surface of the spheroid are the components of attraction for a particle at the surface As the attracted particle passes outward through the surface, κ, in equations (56), starts with the value zero and increases continuously in such a manner that it always fulfills equation (46) Therefore equations (59), having no discontinuity as the attracted particle reaches the surface, hold when α, β, γ fulfill the equation of the ellipsoid

When e is small, as in the case of planets, equations (59) are convenient when expanded as power series in e Substituting the expansions

$$\begin{cases} \sqrt{1-e^2} = 1 - \dfrac{e}{2} - \dfrac{e^4}{8} - \quad , \\[2ex] \sin^{-1} e = e + \dfrac{e^3}{6} + \dfrac{3e^5}{40} + \quad , \\[2ex] \sqrt{1-e^2}\tan^{-1}\left(\dfrac{e}{\sqrt{1-e^2}}\right) = \sqrt{1-e^2}\left(\dfrac{e}{\sqrt{1-e^2}}\right) - \dfrac{\sqrt{1-e^2}}{3}\left(\dfrac{e}{\sqrt{1-e^2}}\right)^3 \\[2ex] \qquad\qquad + \dfrac{\sqrt{1-e}}{5}\left(\dfrac{e}{\sqrt{1-e^2}}\right)^5 - \\[2ex] \qquad = e - \dfrac{e^3}{3} - \dfrac{2}{15}e \quad , \end{cases}$$

in (59), it is found that

$$(60)\quad \begin{cases} \dfrac{X}{\alpha} = \dfrac{Y}{\beta} = -\tfrac{4}{3}\pi\sigma k^2\left(1 - \tfrac{1}{5}e^2 \quad\right), \\[2ex] \dfrac{Z}{\gamma} = -\tfrac{4}{3}\pi\sigma k\ \left(1 + \tfrac{2}{5}e \quad\right) \end{cases}$$

The mass of the spheroid is

$$M = \tfrac{4}{3}\pi\sigma a\, c = \tfrac{4}{3}\pi\sigma a^3\sqrt{1-e^2}$$

The radius of a sphere having equal mass is defined by the equation

$$M = \tfrac{4}{3}\pi\sigma R^3 = \tfrac{4}{3}\pi\sigma a^3 \sqrt{1-e^2},$$

whence

$$R = a(1-e^2)^{\frac{1}{6}}$$

The attraction of this sphere for a particle upon its surface is given by the equation

$$F = -\frac{k^2 M}{R^2} = -\tfrac{4}{3}\pi\sigma k^2 a(1-e^2)^{\frac{1}{6}} \tag{61}$$

When the attracted particle is at the equator of the spheroid $\sqrt{\alpha^2+\beta^2} = a$, hence the ratio of the attraction of the spheroid for a particle at its equator to that of an equal sphere for a particle upon its surface is

$$\frac{\sqrt{X^2+Y^2}}{F} = \frac{(1-\frac{1}{5}e^2 \quad)}{(1-e^2)^{\frac{1}{6}}} = 1 - \frac{e^2}{30}$$

This is less than unity when e is small, therefore the attraction of the spheroid for a particle at its equator is less than that of a sphere of equal mass and volume

When the attracted particle is at the pole of the spheroid $\gamma = c = a\sqrt{1-e^2}$, hence

$$\frac{Z}{F} = \sqrt{1-e^2}\,\frac{(1+\frac{2}{5}e^2 \quad)}{(1-e^2)^{\frac{1}{6}}} = 1 + \frac{e^2}{15}$$

This is greater than unity when e is small, therefore the attraction of the spheroid for a particle at its pole is greater than that of a sphere of equal mass and volume

There is some place between the equator and pole at which the attractions are just equal The latitude of this place will now be found The coordinates of the particle must fulfill the equation of the spheroid, therefore

$$f(\alpha, \beta, \gamma) \equiv \frac{\alpha^2+\beta^2}{a^2} + \frac{\gamma^2}{c^2} - 1 = 0 \tag{62}$$

The direction cosines of the normal to the surface at the point (α, β, γ) are*

* The equations of the tangent plane at the point (x_0, y_0, z_0) are

$$\frac{x_0 - x}{\frac{\partial f}{\partial x}} = \frac{y_0 - y}{\frac{\partial f}{\partial y}} = \frac{z_0 - z}{\frac{\partial f}{\partial z}},$$

(Jordan's *Cours d'Analyse*, vol I p 501) Therefore the direction cosines of a line perpendicular to this plane are those given at the top of page 126

$$\frac{\dfrac{\partial f}{\partial \alpha}}{\sqrt{\left(\dfrac{\partial f}{\partial \alpha}\right)^2 + \left(\dfrac{\partial f}{\partial \beta}\right)^2 + \left(\dfrac{\partial f}{\partial \gamma}\right)^2}}, \quad \frac{\dfrac{\partial f}{\partial \beta}}{\sqrt{\left(\dfrac{\partial f}{\partial \alpha}\right)^2 + \left(\dfrac{\partial f}{\partial \beta}\right) + \left(\dfrac{\partial f}{\partial \gamma}\right)^2}},$$

$$\frac{\dfrac{\partial f}{\partial \gamma}}{\sqrt{\left(\dfrac{\partial f}{\partial \alpha}\right) + \left(\dfrac{\partial f}{\partial \beta}\right) + \left(\dfrac{\partial f}{\partial \gamma}\right)^2}}$$

The last is the cosine of the angle between the normal at (α, β, γ) and the z-axis, and is, therefore, the sine of the latitude, which will be represented by ϕ Hence, it follows from (62) that

$$(63) \qquad \sin \phi = \frac{\dfrac{\partial f}{\partial \gamma}}{\sqrt{\left(\dfrac{\partial f}{\partial \alpha}\right)^2 + \left(\dfrac{\partial f}{\partial \beta}\right)^2 + \left(\dfrac{\partial f}{\partial \gamma}\right)^2}} \equiv \frac{\gamma}{\sqrt{(\alpha^2 + \beta^2)(1 - e^2) + \gamma^2}}$$

From (62) and (63) it is found that

$$(64) \qquad \begin{cases} \alpha^2 + \beta^2 = \dfrac{a \cos^2 \phi}{1 - e \sin^2 \phi} = a^2 \cos^2 \phi \{1 + e^2 \sin^2 \phi + \quad \}, \\ \gamma^2 = \dfrac{a^2 (1 - e^2) \sin \phi}{1 - e \sin^2 \phi} = a^2 \sin^2 \phi \{1 - e^2 (1 + \cos \phi) \quad \} \end{cases}$$

Let G represent the whole attraction of the spheroid, or, from (60) and (64),

$$\begin{aligned} G &= -\sqrt{X^2 + Y^2 + Z^2} \\ &= -\tfrac{4}{3}\pi\sigma k^2 \sqrt{(1 - \tfrac{1}{5}e^2 \quad)^2 (\alpha^2 + \beta^2) + (1 + \tfrac{2}{5}e \quad)^2 \gamma^2} \\ &= -\tfrac{4}{3}\pi\sigma k^2 a \left\{1 - \frac{e^2}{10}(1 + \cos^2 \phi) \quad \right\} \end{aligned}$$

The ratio of this expression to that for the attraction of a sphere of equal mass and volume, given by (61), is

$$(65) \qquad \frac{G}{F} = \frac{1 - \dfrac{e^2}{10}(1 + \cos^2 \phi)}{(1 - e\,)^{\frac{1}{6}}} = 1 - \frac{e^2 (1 - 3 \sin^2 \phi)}{30}$$

This becomes equal to unity up to terms of the fourth order in e when $3 \sin \phi = 1$, from which

$$\phi = 35 \;\; 15' \; 52''$$

Let r represent the radius of the spheroid, then

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\psi},$$

where ψ is the angle between the radius and the plane of the equator Since this angle differs from ϕ only by terms of the second and higher orders in e, it follows that, with the degree of approximation employed,

$$r^2 = \frac{a^2(1-e^2)}{1-e^2\cos^2\phi} = a^2(1-e^2)(1+e^2\cos^2\phi + \quad)$$

When $\phi = 35°\ 15'\ 52''$,

$$r^2 = a^2\left(1 - \frac{e^2}{3} \quad \right)$$

The radius of a sphere of equal volume has been found to be given by the equation

$$R^2 = a^2(1-e^2)^{\frac{1}{3}} = a^2\left(1 - \frac{e^2}{3} \quad \right),$$

which is seen to be equal to the radius of the spheroid up to terms of the second order inclusive in the eccentricity Therefore, in the case of an oblate spheroid of small eccentricity the intensity of the attraction is sensibly the same for a particle on its surface in latitude 35° 15′ 52″ as that of a sphere of equal mass and volume for a particle on its surface, or, because of the equality of R and r, a spheroid of small eccentricity attracts a particle on its surface in latitude 35° 15′ 52″ with sensibly the same intensity it would if its mass were all at its center

XIII PROBLEMS

1 Show that Ivory's lemma may be applied when the attraction varies as any power of the distance

2 Show why Ivory's method cannot be used to find the potential of a solid ellipsoid upon an exterior particle when it is known for an interior particle

3 Find the potential of a thin ellipsoidal shell contained between two similar ellipsoids upon an interior particle *Hint* It has been proved (Art 79) that the resultant attraction is zero at all interior points, therefore the potential is constant and it is sufficient to find it for the center Let the semi axes of the two surfaces be a, b, c and $(1+\mu)a$, $(1+\mu)b$, $(1+\mu)c$, then

the distance between the two surfaces measured along the radius from the center will be $\mu\rho$ Therefore

$$V=\sigma\mu\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{\rho^3\cos\phi\,d\phi\,d\theta}{\rho}=\sigma\mu\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{\cos\phi\,d\phi\,d\theta}{\frac{\cos^2\phi\cos^2\theta}{a^2}+\frac{\cos^2\phi\sin^2\theta}{b^2}+\frac{\sin^2\phi}{c^2}}$$

$$=2\pi\sigma\mu abc\int_0^{\infty}\frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}$$

4 Show that in the case of two thin confocal shells similar elements of mass at points which correspond according to the definition (47) are proportional to the products of the three axes of the respective ellipsoids Then show, using problem 3 and Ivory's method, that the potential of an ellipsoidal shell upon an exterior particle is

$$V'=2\pi\sigma\mu abc\int_0^{\infty}\frac{ds'}{\sqrt{(a'^2+s')(b'^2+s')(c'^2+s')}}=2\pi\sigma abc\int_{\kappa}^{\infty}\frac{ds}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}$$

5 Prove that the level surfaces of thin homogeneous ellipsoids are confocal ellipsoids What are the lines of force which are orthogonal to these surfaces?

6 Discuss the form of level surfaces when they are entirely exterior to homogeneous solid ellipsoids

HISTORICAL SKETCH AND BIBLIOGRAPHY

The attractions of bodies were first investigated by Newton His results are given in the *Principia*, Book I, Secs XII and XIII, and are derived by synthetic processes similar to those used in the first part of this chapter The problem of the attraction of ellipsoids has been the subject of many memoirs, and the case in which they are homogeneous was completely solved early in the nineteenth century Among the important papers are those by Stirling, 1735, *Phil Trans*, by Euler, 1738, *Petersburg*, by Lagrange, 1773 and 1775, *Coll Works*, vol III p 619, by Laplace, 1782, *Méc Cél*, vol II, by Ivory, 1809—1828, *Phil Trans*, by Legendre, 1811, *Mém de l'Inst de France*, vol XI, by Gauss, *Coll Works*, vol V, by Rodriguez, 1816, *Corres sur l'Ecole Poly*, vol III, by Poisson, 1829, *Conn des Temps*, by Green, 1835, *Math Papers*, vol VIII, Chasles, 1837—1846, *Jour l'École Poly* and *Mém des Savants Étrangers*, vol IX, MacCullagh, 1847, *Dublin Proc*, vol III, Lejeune Dirichlet, *Journal de Liouville*, vol IV, and *Crelle*, vol XXXII

The earlier papers were devoted for the most part to the attractions of homogeneous ellipsoids of revolution upon particles in particular positions, as on the axis Lagrange gave the general solution for the attractions of

general homogeneous ellipsoids upon interior particles This was extended by Ivory and Maclaurin (with Laplace's generalizations) to exterior particles Ivory's theorem has been extended in a most interesting manner by Darboux in Note XVI to the second volume of the *Mécanique* of Despeyrous Laplace proved that the potential for an exterior particle fulfills the partial differential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

and determined V by the condition that it must be a function satisfying this equation This is a process of great generality, and is relatively simple except in the trivial cases This has been made the starting point of most of the investigations of the latter part of the last century, especially where the attracting bodies are not homogeneous In a paper on Electricity and Magnetism in 1828 Green introduced the term *potential function* for V, and discussed many of its mathematical properties Green's memoir remained nearly unknown until about 1846, and in the meantime many of his theorems had been rediscovered by Chasles, Gauss, Sturm, and Thomson One of Green's theorems has been found an extremely useful application, when the independent variables are two in number, in the Theory of Functions

Poisson showed that the potential function for an interior particle fulfills the partial differential equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-4\pi\sigma$$

Chasles gave a synthetic proof of the theorems regarding the attractions of homogeneous ellipsoids in *Mémoires des Savants Étrangers*, vol IX, and Lejeune Dirichlet embraced in a most elegant manner in one discussion the case of both interior and exterior points by using a discontinuous factor (*Liouville's Journal*, vol IV)

Many of the later investigations have been for heterogeneous bodies, or for forces other than the Newtonian gravitation

Among the books treating the subject of attractions and potential may be mentioned Thomson and Tait's *Natural Philosophy*, part II, Neumann's *Potential*, Poincaré's *Potential*, Routh's *Analytical Statics*, vol II, and Tisserand's *Mécanique Céleste*, vol II The last mentioned develops most fully the astronomical applications and should be used in further reading

CHAPTER V

THE PROBLEM OF TWO BODIES

83 Equations of Motion It will be assumed in this chapter that the two bodies are spheres and homogeneous in concentric layers Then, in accordance with the results obtained in Art 69, they will attract each other with a force which is proportional to the product of their masses and varies inversely as the square of the distance of their centers apart

Let m_1 and m_2 represent their masses, and $m_1 + m_2 = M_{1,2}$ Choose an arbitrary system of rectangular axes in space and let the coordinates of m_1 and m_2 referred to it be respectively ξ_1, η_1, ζ_1, and ξ_2, η_2, ζ_2 Let the distance between m_1 and m_2 be denoted by $r_{1,2}$ then, the differential equations of motion are

$$(1)\quad \begin{cases} m_1 \dfrac{d^2\xi_1}{dt^2} = -k^2 m_1 m_2 \dfrac{(\xi_1 - \xi_2)}{r^3}, \\ m_1 \dfrac{d^2\eta_1}{dt^2} = -k^2 m_1 m_2 \dfrac{(\eta_1 - \eta_2)}{r^3}, \\ m_1 \dfrac{d^2\zeta_1}{dt^2} = -k^2 m_1 m_2 \dfrac{(\zeta_1 - \zeta_2)}{r^3}, \\ m_2 \dfrac{d^2\xi_2}{dt^2} = -k^2 m_2 m_1 \dfrac{(\xi_2 - \xi_1)}{r^3}, \\ m_2 \dfrac{d^2\eta_2}{dt^2} = -k^2 m_2 m_1 \dfrac{(\eta_2 - \eta_1)}{r^3}, \\ m_2 \dfrac{d^2\zeta_2}{dt^2} = -k^2 m_2 m_1 \dfrac{(\zeta_2 - \zeta_1)}{r^3} \end{cases}$$

In order to solve these six simultaneous equations of the second order twelve integrals must be found They will introduce twelve arbitrary constants of integration which may be determined in any

particular case by the three initial coordinates and the three components of the initial velocity of each of the bodies

84 The Motion of the Center of Mass Adding the first and fourth equations, the second and fifth, and the third and sixth, it is found that

$$\left\{\begin{aligned} m_1 \frac{d^2\xi_1}{dt^2} + m_2 \frac{d^2\xi_2}{dt^2} &= 0, \\ m_1 \frac{d^2\eta_1}{dt^2} + m_2 \frac{d^2\eta_2}{dt^2} &= 0, \\ m_1 \frac{d^2\zeta_1}{dt^2} + m_2 \frac{d^2\zeta_2}{dt^2} &= 0 \end{aligned}\right.$$

These equations are immediately integrable, and give

$$(2)\qquad \left\{\begin{aligned} m_1 \frac{d\xi_1}{dt} + m_2 \frac{d\xi_2}{dt} &= \alpha_1, \\ m_1 \frac{d\eta_1}{dt} + m_2 \frac{d\eta_2}{dt} &= \beta_1, \\ m_1 \frac{d\zeta_1}{dt} + m_2 \frac{d\zeta_2}{dt} &= \gamma_1 \end{aligned}\right.$$

Integrating again,

$$(3)\qquad \left\{\begin{aligned} m_1\xi_1 + m_2\xi_2 &= \alpha_1 t + \alpha_2, \\ m_1\eta_1 + m_2\eta &= \beta_1 t + \beta_2, \\ m_1\zeta_1 + m_2\zeta_2 &= \gamma_1 t + \gamma_2 \end{aligned}\right.$$

Thus, six of the twelve integrals are found, the arbitrary constants of integration being α_1, α_2, β_1, β_2, γ_1, γ_2 Let $\bar{\xi}$, $\bar{\eta}$, and $\bar{\zeta}$ be the coordinates of the center of mass of the system, then it follows from Art 19 and equations (3) that

$$(4)\qquad \left\{\begin{aligned} M_{1,2}\,\bar{\xi} &= m_1\xi_1 + m_2\xi_2 = \alpha_1 t + \alpha_2, \\ M_{1,2}\,\bar{\eta} &= m_1\eta_1 + m_2\eta_2 = \beta_1 t + \beta_2, \\ M_{1,2}\,\bar{\zeta} &= m_1\zeta_1 + m_2\zeta_2 = \gamma_1 t + \gamma_2 \end{aligned}\right.$$

From these equations it follows that the coordinates increase directly as the time, and, therefore, that the center of mass moves with uniform velocity Or, taking their derivatives, squaring, and adding, it is found that

$$M_{1,2}^2\left\{\left(\frac{d\bar{\xi}}{dt}\right)^2 + \left(\frac{d\bar{\eta}}{dt}\right)^2 + \left(\frac{d\bar{\zeta}}{dt}\right)^2\right\} = \alpha_1^2 + \beta_1^2 + \gamma_1^2,$$

whence

$$\bar{V} = \frac{\sqrt{\alpha_1^2 + \beta_1^2 + \gamma_1^2}}{M_{1,2}},$$

where $\overline{V}$ represents the speed with which the center of mass moves The speed is therefore constant

Eliminating t from (4), it follows that

$$\frac{M_{1,2}\overline{\xi}-\alpha}{\alpha_1}=\frac{M_1\ \overline{\eta}-\beta_2}{\beta_1}=\frac{M_1\ \overline{\zeta}-\gamma}{\gamma_1}$$

The coordinates of the center of mass fulfill these relations which are the symmetrical equations of a straight line in space, therefore, *the center of mass moves in a straight line with constant speed*

85 The Equations for Relative Motion Take a new system of axes parallel to the old, but with the origin at the center of mass of the two bodies Let the coordinates of m_1 and m_2 referred to this new system be x_1, y_1, z_1, and x_2, y_2, z_2, respectively They are related to the old coordinates by the equations

$$(5)\qquad \begin{cases} x_1=\xi_1-\overline{\xi}, & x_2=\xi\ -\overline{\xi}, \\ y_1=\eta_1-\overline{\eta}, & y_2=\eta\ -\overline{\eta}, \\ z_1=\zeta_1-\overline{\zeta}, & z_2=\zeta_2-\overline{\zeta} \end{cases}$$

Substituting in (1), the differential equations of motion in the new variables are found to be

$$(6)\qquad \begin{cases} m_1\dfrac{d\ x_1}{dt}=-k\ m_1 m\ \dfrac{(x_1-x)}{r^3}, \\ m_1\dfrac{d\ y_1}{dt}=-k\ m_1 m\ \dfrac{(y_1-y)}{r^3}, \\ m_1\dfrac{d^2 z_1}{dt^2}=-k\ m_1 m_2\ \dfrac{(z_1-z_2)}{r^3}, \\ m_2\dfrac{d\ x_2}{dt}=-k\ m\ m_1\dfrac{(x_2-x_1)}{r^3}, \\ m_2\dfrac{d\ y_2}{dt}=-k\ m\ m_1\dfrac{(y\ -y_1)}{r^3}, \\ m_2\dfrac{d\ z_2}{dt^2}=-k^2 m\ m_1\dfrac{(z\ -z_1)}{r^3}, \end{cases}$$

which are of the same form as the equations for absolute motion

The coordinates of the center of mass are given by equations (4), therefore if x_1, y_1, , z_2 were known, and if the constants α_1, α_2, β_1, β_2, γ_1, and γ were known, the absolute positions in space could be found But, since there is no way of determining these constants, the

problem of relative motion, as expressed in (6), is all that can be solved

Since the new origin is at the center of mass, the coordinates are related by the equations

$$(7)\qquad \begin{cases} m_1x_1 + m_2x_2 = 0, \\ m_1y_1 + m_2y_2 = 0, \\ m_1z_1 + m_2z_2 = 0 \end{cases}$$

Therefore, when the coordinates of one body with respect to the center of mass of both are known the coordinates of the other are given by (7), and if the constants in (3) could be determined, the coordinates of both with respect to fixed axes in space could be found By means of (7) the coordinates of the second body might be eliminated from the first three equations of (6), after which they might be solved independently of the last three for x_1, y_1, and z_1 It will be preferable, however, to solve for the position of one body with respect to the other From (7) it is easily found that

$$(8)\qquad \begin{cases} \dfrac{x_2}{x_1 - x_2} = \dfrac{-m_1}{m_1 + m_2} = \dfrac{-m_1}{M_{1,2}}, \\[2ex] \dfrac{x_1}{x_1 - x_2} = \dfrac{m_2}{M_{1,2}}, \end{cases}$$

and similar equations in y and z

Let the coordinates of m_1 with respect to m_2, as origin, be x, y, z, then

$$\begin{cases} x = x_1 - x_2, \\ y = y_1 - y_2, \\ z = z_1 - z_2 \end{cases}$$

Therefore (8) gives

$$\begin{cases} x_1 = \dfrac{m_2 x}{M_{1,2}}, & x_2 = \dfrac{-m_1 x}{M_{1,2}}, \text{ and similarly,} \\[2ex] y_1 = \dfrac{m_2 y}{M_{1,2}}, & y_2 = \dfrac{-m_1 y}{M_{1,2}}, \\[2ex] z_1 = \dfrac{m_2 z}{M_{1,2}}, & z_2 = \dfrac{-m_1 z}{M_{1,2}} \end{cases}$$

By these equations the coordinates of the bodies with respect to their center of mass are defined when their relative positions are

known. Substituting in (6), the differential equations of motion of m_1 referred to m_2 are found to be

$$
(9) \qquad \begin{cases}
\dfrac{d^2 x}{dt^2} = -k^2 M_{1,2} \dfrac{x}{r^3}, \\[2ex]
\dfrac{d^2 y}{dt^2} = -k^2 M_{1,2} \dfrac{y}{r^3}, \\[2ex]
\dfrac{d^2 z}{dt^2} = -k^2 M_{1,2} \dfrac{z}{r^3}.
\end{cases}
$$

The problem is now of the sixth order, having been reduced from the twelfth by means of the six integrals (2) and (3). The six new constants of integration which will be introduced in integrating equations (9) will be determined by the three initial coordinates, and the three projections of the initial velocity of m_1 with respect to m_2.

86. The Integrals of Areas. Multiply the first equation of (9) by $-y$, and the second by x, and add; the result is

$$
\begin{cases}
x \dfrac{d^2 y}{dt^2} - y \dfrac{d^2 x}{dt^2} = 0, \text{ and similarly,} \\[2ex]
y \dfrac{d^2 z}{dt^2} - z \dfrac{d^2 y}{dt^2} = 0, \\[2ex]
z \dfrac{d^2 x}{dt^2} - x \dfrac{d^2 z}{dt^2} = 0.
\end{cases}
$$

The integrals of these equations are

$$
(10) \qquad \begin{cases}
x \dfrac{dy}{dt} - y \dfrac{dx}{dt} = a_1, \\[2ex]
y \dfrac{dz}{dt} - z \dfrac{dy}{dt} = a_2, \\[2ex]
z \dfrac{dx}{dt} - x \dfrac{dz}{dt} = a_3.
\end{cases}
$$

It follows from Art. 16 that a_1, a_2, a_3 are the projections of twice the areal velocity upon the xy, yz, and zx planes respectively. Multiplying equations (10) by z, x, and y respectively, and adding, it is found that

$$
(11) \qquad a_1 z + a_2 x + a_3 y = 0.
$$

This is the equation of a plane passing through the origin, and it follows from its derivation that the coordinates of m_1 always fulfill it;

therefore, *the motion of one body with respect to the other is in a plane passing through the center of the other*

The constants a_1, a_2, and a_3 determine the position of the plane of the orbit with respect to the axes of reference Equation (11) becomes in polar coordinates

$$a_1 \sin\phi + a_2 \cos\phi \cos\theta + a_3 \cos\phi \sin\theta = 0 \tag{12}$$

Let Ω represent the angle between the positive end of the x-axis and the line of intersection of the xy-plane and the plane of the orbit, let i represent the inclination of the two planes Then, when $\phi = 0$, $\theta = \Omega + n\pi$, where n is any integer, and when $\theta = \Omega + \frac{\pi}{2}$, $\phi = i$ Making these substitutions in (12), it is found that

$$\tag{13} \begin{cases} \tan\Omega = -\dfrac{a_2}{a_3}, \\[2ex] \tan i = \dfrac{\sqrt{a_2^2 + a_3^2}}{a_1} \end{cases}$$

Since the projections of the areal velocity upon the three fundamental planes are constants the areal velocity in the plane of the orbit is also constant Let it be represented by c_1 It is defined by the equation

$$c_1 = \sqrt{a_1^2 + a_2^2 + a_3^2} \tag{14}$$

Solving (13) and (14), it is found that

$$\tag{15} \begin{cases} a_1 = c_1 \cos i, \\ a_2 = c_1 \sin i \sin\Omega, \\ a_3 = -c_1 \sin i \cos\Omega \end{cases}$$

87 Problem in the Plane Since the orbit lies in a known plane, the coordinate axes may be chosen so that the x and y-axes lie in this plane If the coordinates are represented by x and y as before, the differential equations of motion are

$$\tag{16} \begin{cases} \dfrac{d^2x}{dt^2} = -k^2 M_{1\ 2} \dfrac{x}{r^3}, \\[2ex] \dfrac{d^2y}{dt^2} = -k^2 M_{1\ 2} \dfrac{y}{r^3} \end{cases}$$

The problem is now of the fourth order instead of the sixth as it was in (9), having been reduced by means of the integrals (10) It will be observed that, since the position of the plane is defined by the

two elements ☊ and i, or by the ratios of a_1, a_2, and a_3 in (11), only two of the arbitrary constants were involved in the reduction This problem might be solved by deriving the differential equation of the orbit as in Art 54 and integrating as in Art 62, the last integral being derived from the integral of areas, but, it is preferable to obtain the results directly by the method which is usually employed in Celestial Mechanics

Equations (16) give

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0$$

The integral of this equation is

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c_1,$$

which becomes in polar coordinates

$$r^2\frac{d\theta}{dt} = c_1 \tag{17}$$

Let A represent the area swept over by the radius vector r, then

$$2\frac{dA}{dt} = r^2\frac{d\theta}{dt} = c_1,$$

whence

$$2A = c_1 t + c, \tag{18}$$

from which it follows that the areas swept over by the radius vector are proportional to the times in which they are described

Multiplying (16) by $2\frac{dx}{dt}$ and $2\frac{dy}{dt}$ respectively, and adding, the result is

$$2\frac{d^2x}{dt^2}\frac{dx}{dt} + 2\frac{d^2y}{dt^2}\frac{dy}{dt} = -2\frac{k^2M_1}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = -2k^2M_1\frac{1}{r^2}\frac{dr}{dt}$$

The integral of this equation is

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{2k^2M_1}{r} + c_3 \tag{19}$$

Transforming the left member to polar coordinates, this equation becomes

$$\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{2k^2M_1}{r} + c_3$$

But

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt},$$

therefore

$$\left(\frac{d\theta}{dt}\right)^2\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} = \frac{2k^2M_1}{r} + c_3$$

Eliminating $\frac{d\theta}{dt}$ by means of (17), this equation gives

$$d\theta = \frac{c_1 dr}{r\sqrt{-c_1^2 + 2k^2 M_{1,2} r + c_3 r^2}},$$

which may be written

$$(20) \qquad d\theta = \frac{-d\left(\frac{c_1}{r}\right)}{\sqrt{c_3 + \frac{k^4 M_{1,2}^2}{c_1^2} - \left(\frac{k^2 M_{1,2}}{c_1} - \frac{c_1}{r}\right)^2}}$$

Let

$$\begin{cases} c_3 + \dfrac{k^4 M_{1,2}^2}{c_1^2} = B^2, \\ \dfrac{k^2 M_{1,2}}{c_1} - \dfrac{c_1}{r} = -u, \end{cases}$$

in which B must be positive for a real orbit, then (20) becomes

$$d\theta = \frac{-du}{\sqrt{B^2 - u^2}}$$

The integral of this equation is

$$\theta = \cos^{-1}\frac{u}{B} + c_4$$

Expressing this in terms of r and the original constants, it is found that

$$(21) \qquad r = \frac{c_1}{\frac{k^2 M_{1,2}}{c_1} - \sqrt{c_3 + \frac{k^4 M_{1,2}^2}{c_1^2}}\cos(\theta - c_4)},$$

which is the polar equation of a conic section with the origin at one of its foci

88 The Elements in Terms of the Constants of Integration The node and inclination are expressed in terms of the constants of integration by (13)

The ordinary equation of a conic section with the origin at the right-hand focus is

$$r = \frac{p}{1 + e\cos(\theta - \omega)},$$

where p is the semi-parameter, and ω the angle between the polar axis

and the major axis of the conic Comparing this equation with (21), it is found that

$$
(22) \qquad \begin{cases} p = \dfrac{c_1^2}{k^2 M_1}, \\ e^2 = 1 + \dfrac{c_1^2 c_3}{k^4 M_1^2}, \\ \omega = c_4 - \pi, \\ c_1 = k\sqrt{M_1 p} \\ c_3 = -\dfrac{k^2(1-e^2)}{p} M_1 \end{cases}
$$

When $e < 1$, the orbit is an ellipse and $p = a(1 - e^2)$, where a is the major semi-axis, when $e^2 = 1$, the orbit is a parabola and $p = 2q$ where q is the distance from the origin to the vertex of the parabola, and when $e > 1$, the orbit is an hyperbola and $p = a(e^2 - 1)$

Let A_0 represent the area described at the time the body passes perihelion*, then the time of perihelion passage is found from (18) to be

$$
(23) \qquad T = \frac{2A_0 - c_2}{c_1}
$$

In place of this element it is generally more convenient to use the mean longitude at the epoch t_0

This completes the determination of the elements in terms of the constants of integration The constants of integration are defined in terms of initial coordinates and components of velocity by the equations where they first occur, viz (10), (17), (18), (19), and (21)

89 Properties of the Motion Suppose the orbit is an ellipse Then, when the values of the constants of integration given in (22) are substituted in (17) and (19), these equations become

$$
(24) \qquad \begin{cases} r^2 \dfrac{d\theta}{dt} = k\sqrt{M_1 \, a(1-e^2)}, \\ \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = V^2 = k^2 M_1 \left(\dfrac{2}{r} - \dfrac{1}{a}\right), \end{cases}
$$

where V is the speed in the orbit at the distance r from the origin

When the orbit is a circle, $r = a$ and

$$
V_c^2 = \frac{k^2 M_1}{r}
$$

* Unless m is specified to be some body other than the sun the nearest apse will be called the perihelion point

When the orbit is a parabola, $a = \infty$ and

$$V_p^2 = \frac{2k^2 M_{1\,2}}{r}$$

Therefore, at a given distance from the origin the ratio of the speed in a parabolic orbit to that in a circular orbit is

(25) $$V_p : V_c = \sqrt{2} : 1$$

Thus, in the motion of comets around the sun they cross the planets' orbits with velocities about 1.414 times those with which the respective planets move

The speed that a body will acquire in falling from the distance s to the distance r toward the center of force $k^2 M_{1\,2}$ is given by (see Art 35)

$$V^2 = 2k^2 M_{1\,2}\left(\frac{1}{r} - \frac{1}{s}\right)$$

If s is determined by the condition that this shall equal the speed in the orbit, it is found, after equating the right member of this expression to the right member of the second of (24), that

$$s = 2a$$

Therefore, *the speed of a body moving in an ellipse is at every point equal to that which it would acquire in falling from the circumference of a circle, with center at the origin and radius equal to the major axis of the conic, to the ellipse**

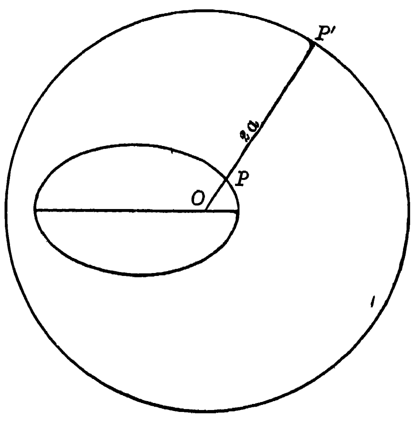

Fig 25

The speed at P in the ellipse is equal to that which would be acquired in falling from P' to P

* Proved by Van der Kolk, *Astronomische Nachrichten*, No 1426

When the body falls from infinity, $s = \infty$ Let U represent the corresponding value of V, then

$$U^2 = \frac{2k^2 M_{1,2}}{r}$$

Eliminating $k^2 M_{1,2}$ from the last of (24) by means of this equation and solving for a, it is found that

$$a = \frac{r}{2}\left(\frac{U^2}{U^2 - V^2}\right) \tag{26}$$

U depends upon the masses of the two bodies and their distance apart, therefore, *the major axis of the conic depends upon the initial distance from the origin, and the initial speed, and not upon the direction of projection*

Let t_1 and t_2 be two epochs, and A_1 and A_2 the corresponding values of the area described by the radius vector Then equation (18) gives

$$2(A_2 - A_1) = (t_2 - t_1)\, c_1$$

Suppose $t_2 - t_1 = P_{1,2}$, the period of revolution, then $2(A_2 - A_1)$ equals twice the area of the ellipse, equals $2\pi ab$ The expression for the period, found by substituting the value of c_1 given in (22) and solving, is

$$P_{1,2} = \frac{2\pi a^{\frac{3}{2}}}{k\sqrt{M_{1,2}}} \tag{27}$$

From this equation it follows that the period is independent of every element except the major axis, or, because of (26), the period depends

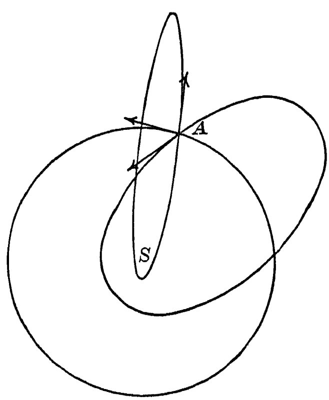

Fig 26

only upon the initial distance from the origin and the initial speed, and not upon the direction of projection The major semi-axis will be

called the *mean distance*, although it must be understood that it is *not* the *average* distance

The three orbits drawn in the figure have the same length of major axis and are consequently described in the same time The speed of projection from A is the same in each case, the differences in the shapes and positions resulting from the different directions of projection

If the two systems m_1, m_2, and m_2, m_3 are considered, and the ratio of their periods is taken, it is found that

$$\frac{P^2_{1\,2}}{P^2_{3\,2}} = \frac{a^3_{1\,2}}{a^3_{3\,2}} \frac{M_{3\,2}}{M_{1\,2}}$$

If the two systems are composed of the sun and two planets respectively, then $M_{1\,2}$ and $M_{3\,2}$ are very nearly equal because the masses of the planets are exceedingly small compared to that of the sun Therefore, this equation becomes very nearly

$$\frac{P^2_{1\,2}}{P^2_{3\,2}} = \frac{a^3_{1\,2}}{a^3_{3\,2}},$$

or, *the squares of the periodic times of the planets are proportional to the cubes of their mean distances* This is Kepler's third law

It is to be observed that, in taking the ratios of the periods, it was assumed that k has the same value for the different planets, or, that the sun's acceleration of the various planets would be the same at unit distance On the other hand, it follows from the last equation, which Kepler established directly by observations, that k has the same value for the various planets

90 Selection of Units and the Determination of the Constant k When the units of time, mass, and distance are chosen k may be determined from (27) It is evident that they may all be taken arbitrarily, but it will be convenient to employ those units in which astronomical problems are most frequently treated The mean solar day will be taken as the unit of time, the mass of the sun will be taken as the unit of mass, and the major semi-axis of the earth's orbit will be taken as the unit of distance When these units are employed the k determined by them is called the Gaussian constant, having been defined in this way by Gauss in the *Theoria Motus*, Art 1

Let m represent the mass of the sun and m_1 that of the earth and moon together, then it has been found from observation that in these units

$$(28) \qquad \begin{cases} m_1 = \dfrac{m_2}{354710} = \dfrac{1}{354710}, \\ P_1 \quad = 365\ 2563835 \end{cases}$$

Substituting in (27), it is found that

$$(29) \qquad \begin{cases} k = \dfrac{2\pi}{P_{1\ 2}\sqrt{1+m_1}} = 0\ 01720209895, \\ \log k = 8\ 2355814414 - 10 \end{cases}$$

Since m_1 is very small $k = \dfrac{2\pi}{P_{1\ 2}}$ nearly, and is, therefore, nearly the mean daily motion of the earth in its orbit, or about $\frac{1}{60}$ The mean daily motion of a planet whose mass is m_i is $\dfrac{2\pi}{P_i}$, and is usually designated by n_i This is found from (27) to be

$$(30) \qquad n_i = \frac{k\sqrt{1+m_i}}{a^{\frac{3}{2}}_i}$$

The period of the earth's revolution around the sun and its mean distance were not known with perfect exactness at the time of Gauss, nor are they yet, and it is clear that the value of k varies with the different determinations of these quantities If astronomers held strictly to the definitions of the units given above it would be necessary to recompute those tables which depend upon k every time an improvement in the values of the constants is made These inconveniences are avoided by keeping the numerical value of k that which Gauss determined, and choosing the unit of distance so that (27) will always be fulfilled

If the mean distance between two bodies is taken as the unit of distance and the sum of their masses as the unit of mass, and if the unit of time is taken so that k equals unity, then the units form what is called a *canonical system* Since $M_1 = 1$ and $k^2 = 1$ in this system, and from (30) $n = 1$, the equations become somewhat simplified and are advantageous in purely theoretical investigations

XIV PROBLEMS

1 Derive the theorem of the uniform rectilinear motion of the center of gravity directly from Newton's laws of motion

2 Derive equations (9) from equations (1) by a direct transfer of the origin to the center of m_2

3 Find the differential equations of motion in polar coördinates

Ans
$$\begin{cases} \dfrac{d^2r}{dt^2} = r\left(\dfrac{d\theta}{dt}\right)^2 - \dfrac{k^2M_{1\,2}}{r^2}, \\ \dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = 0 \end{cases}$$

4 Integrate the equations of problem 3 and interpret the constants of integration

5 The earth moves in its orbit, which may be assumed to be circular, with a speed of 18 5 miles per second Suppose the meteors approach the sun in parabolas, between what limits will be their relative speed when they strike into the earth's atmosphere?

Ans 7 66 to 44 66 miles per second. (The Nov 14 meteors meet the earth and have a relative speed near the upper limit, the Nov 27 meteors overtake the earth and have a relative speed near the lower limit)

6 Find the average length of the radius vector of an ellipse in terms of a and e, taking the time as the independent variable

Ans
$$\text{Average } r = \frac{\int r\,dt}{\int dt} = a\left(1 + \frac{e^2}{2}\right)$$

7 Find the average length of the radius vector of an ellipse, taking the angle as the independent variable

Ans
$$\text{Average } r = \frac{\int r\,d\theta}{\int d\theta} = \frac{2\pi a\sqrt{1-e^2}}{2\pi} = b$$

8 Prove that the amount of heat received from the sun by the planets per unit area is proportional to the reciprocals of the products of the major and minor axes of their orbits

9 If a planet is projected from a given point with a given velocity, find the locus of, (*a*) perihelion points, (*b*) aphelion points, (*c*) center of ellipses, (*d*) ends of minor axes

10 If a planet is projected from a given point in a given direction, find the loci of the same points as in problem 9, and express the coördinates of these points in terms of the initial values of the coördinates and the components of velocity

91 Position in Parabolic Orbits Having found the curves in which the bodies move, it remains to find their positions in their orbits at any given epoch The case of the parabolic orbit being the simplest will be considered first, and it will be supposed, to fix the ideas, that the motion is that of a comet with respect to the sun Since the masses of the comets are negligible $M_1 = 1$, and (17) becomes

$$(31) \qquad r^2 \frac{d\theta}{dt} = k\sqrt{p} = k\sqrt{2q}$$

When the polar angle is counted from the vertex of the parabola it is denoted by v, and is called the *true anomaly* Then

$$\begin{cases} \dfrac{d\theta}{dt} = \dfrac{dv}{dt}, \\ r = \dfrac{p}{1+\cos v} = q \sec^2 \dfrac{v}{2} \end{cases}$$

Hence, (31) gives

$$\frac{\sqrt{2}k}{q^{\frac{3}{2}}} dt = \sec^4 \frac{v}{2}\, dv = \left(\sec^2 \frac{v}{2} + \sec^2 \frac{v}{2} \tan^2 \frac{v}{2}\right) dv$$

The integral of this expression is

$$(32) \qquad \tan \frac{v}{2} + \frac{1}{3} \tan^3 \frac{v}{2} = \frac{k(t-T)}{\sqrt{2}q^{\frac{3}{2}}},$$

where T is the time of perihelion passage This is a cubic equation in $\tan \frac{v}{2}$ Taking the right member to the left side it is seen that for $t - T > 0$, the function is negative when $v = 0$, and that it increases continually with v until it equals infinity, for $v = 180^\circ$ Therefore there is but one real solution, which is positive For $t - T < 0$ it is seen in a similar manner that there is one real negative solution

Equation (32) may be written

$$25 \tan^3 \frac{v}{2} + 75 \tan \frac{v}{2} = \frac{75k}{\sqrt{2}} \frac{(t-T)}{q^{\frac{3}{2}}} = K \frac{(t-T)}{q^{\frac{3}{2}}}$$

Tables have been constructed giving the value of the right member of this equation for different values of v From these tables v may be found by interpolation when $t - T$ is given, or, conversely, $t - T$ may be found when v is given These tables are known as Barker's, and are VI in Watson's *Theoretical Astronomy*, and IV in Oppolzer's *Bahnbestimmung**

* In Oppolzer's *Bahnbestimmung* the factor 75 is not introduced

In order to find the direct solution of the cubic equation let

$$\tan\frac{v}{2} = 2\cot 2w = \cot w - \tan w\,,$$

whence

$$\tan^3\frac{v}{2} = -3\tan\frac{v}{2} + \cot^3 w - \tan^3 w$$

This substitution reduces (32) to

$$\cot^3 w - \tan^3 w = \frac{3k(t-T)}{\sqrt{2}q^{\frac{3}{2}}}$$

Let

$$\cot w = \sqrt[3]{\cot\frac{s}{2}}\,,$$

whence

$$\cot s = \frac{3k(t-T)}{2^{\frac{3}{2}}q^{\frac{3}{2}}}$$

Therefore the formulas for the computation of $\tan\frac{v}{2}$ are, in the order of their application,

$$(33)\qquad \begin{cases} \cot s = \dfrac{3k(t-T)}{(2q)^{\frac{3}{2}}}, \\[2ex] \cot w = \sqrt[3]{\cot\dfrac{s}{2}}, \\[2ex] \tan\dfrac{v}{2} = 2\cot 2w \end{cases}$$

After v has been found r is determined by the polar equation of the parabola, $r = \frac{p}{1+\cos v} = q\sec^2\frac{v}{2}$

92 Equation involving Two Radii and their Chord Euler's Equation Consider the positions of the comet at the instants t_1 and t_2 Let the corresponding radii be r_1 and r_2, and the chord joining their extremities s Let the corresponding true anomalies be v_1 and v_2 Then it follows that

$$\begin{cases} \dfrac{k(t_1-T)}{\sqrt{2}q^{\frac{3}{2}}} = \tan\dfrac{v_1}{2} + \dfrac{1}{3}\tan^3\dfrac{v_1}{2}, \\[2ex] \dfrac{k(t_2-T)}{\sqrt{2}q^{\frac{3}{2}}} = \tan\dfrac{v_2}{2} + \dfrac{1}{3}\tan^3\dfrac{v_2}{2} \end{cases}$$

The difference of these equations is

$$\frac{k(t_2-t_1)}{\sqrt{2}q^{\frac{3}{2}}} = \tan\frac{v_2}{2} - \tan\frac{v_1}{2} + \frac{1}{3}\left(\tan^3\frac{v_2}{2} - \tan^3\frac{v_1}{2}\right),$$

or,

$$(34)\qquad \frac{3k(t-t_1)}{\sqrt{2}q^{\frac{3}{2}}} = \left(\tan\frac{v_2}{2} - \tan\frac{v_1}{2}\right)\left[3\left(1+\tan\frac{v_1}{2}\tan\frac{v_2}{2}\right) + \left(\tan\frac{v_2}{2} - \tan\frac{v_1}{2}\right)^2\right]$$

The equation for the chord is

$$s^2 = r_1^2 + r_2 \; - 2r_1r_2\cos(v_2 - v_1) = (r_1 + r_2) \; - 4r_1r_2\cos^2\left(\frac{v_2 - v_1}{2}\right)$$

From this equation it is found that

$$(35)\qquad 2\sqrt{r_1r_2}\cos\left(\frac{v_2 - v_1}{2}\right) = \pm\sqrt{(r_1 + r_2 + s)(r_1 + r_2 - s)}$$

The + sign is to be taken if $v_2 - v_1 < \pi$ and the − sign if $v_2 - v_1 > \pi$

It follows from the polar equation of the parabola that

$$r_1 = \frac{q}{\cos^2\frac{v_1}{2}}, \qquad r_2 = \frac{q}{\cos^2\frac{v_2}{2}}$$

These, substituted in (35), give

$$\frac{\cos\left(\frac{v_2 - v_1}{2}\right)}{\cos\frac{v_1}{2}\cos\frac{v_2}{2}} = \pm\frac{\sqrt{(r_1 + r_2 + s)(r_1 + r_2 - s)}}{2q},$$

whence

$$(36)\qquad 1 + \tan\frac{v_1}{2}\tan\frac{v_2}{2} = \pm\frac{\sqrt{(r_1 + r_2 + s)(r_1 + r_2 - s)}}{2q}$$

It also follows from the expressions for r_1 and r_2 that

$$r_1 + r_2 = q\left(2 + \tan^2\frac{v_1}{2} + \tan^2\frac{v_2}{2}\right)$$

The last two equations give

$$\frac{(r_1 + r_2 + s) + (r_1 + r_2 - s) \mp 2\sqrt{(r_1 + r_2 + s)(r_1 + r_2 - s)}}{2q} = \left(\tan\frac{v_2}{2} - \tan\frac{v_1}{2}\right)^2,$$

whence

$$(37)\qquad \frac{\sqrt{r_1 + r_2 + s} \mp \sqrt{r_1 + r \; - s}}{\sqrt{2q}} = \tan\frac{v_2}{2} - \tan\frac{v_1}{2}$$

Equation (34) becomes, as a consequence of (36) and (37),

$$(38)\qquad 6k(t_2 - t_1) = (r_1 + r_2 + s)^{\frac{3}{2}} \mp (r_1 + r_2 - s)^{\frac{3}{2}}$$

This equation is remarkable in that it does not involve q It was discovered by Euler and bears his name It is of the first importance

in the determination of the elements of a parabolic orbit from geocentric observations

There is a corresponding equation, due to Lambert, for elliptic orbits The right member is developed as a power series in $\frac{1}{a}$, the first term constituting the right member of Euler's equation

93 Position in Elliptic Orbits The polar equation of the ellipse and the integral of areas are respectively

$$\begin{cases} r = \dfrac{a(1-e^2)}{1+e\cos v}, \\ r^2 \dfrac{dv}{dt} = k\sqrt{M_{1\,2}\,a(1-e^2)} \end{cases}$$

From the first of these equations it follows that

$$(39) \qquad \begin{cases} \cos v = \dfrac{a(1-e^2)}{er} - \dfrac{1}{e}, \\ dv = \dfrac{a\sqrt{1-e^2}}{r}\,\dfrac{dr}{\sqrt{a^2e^2-(a-r)^2}} \end{cases}$$

Let n represent the mean angular motion, then

$$n = \frac{2\pi}{P_{1\,2}} = \frac{k\sqrt{M_{1\,2}}}{a^{\frac{3}{2}}}$$

As a consequence of these equations the equation of areas becomes

$$(40) \qquad n\,dt = \frac{r}{a}\,\frac{dr}{a\sqrt{a^2e^2-(a-r)^2}}$$

This may be integrated at once, expressing the interval of time in terms of $\frac{a-r}{ae}$, but it is desired to express r in terms of t This might be done by expanding the function into a series in $\frac{a-r}{ae}$ and then inverting the series The first series converges so long as the numerical value of $a-r$ is less than ae, and this inequality holds except at the ends of the major axis

This procedure would be very inconvenient in practice, hence the problem will be solved in another way Let the auxiliary E be introduced by the equation

$$(41) \qquad \begin{cases} ae\cos E = a - r, \\ r = a(1 - e\cos E) \end{cases}$$

This angle E is called the *eccentric anomaly* Then (40) becomes

$$n\,dt = (1 - e\cos E)\,dE,$$

the integral of which is

$$n(t - T) = E - e\sin E$$

$n(t-T)$ is the angle which would have been described by the radius vector if it had moved uniformly with the average rate It is usually denoted by M and is called the *mean anomaly* Therefore

$$(42) \qquad n(t - T) = M = E - e\sin E$$

The M may at once be found when $(t - T)$ is given, after which equation (42) must be solved for E Then r and v may be found by (41) and (39) respectively Equation (42), known as *Kepler's equation*, is transcendental in E, and the solution for this quantity cannot be expressed in a finite number of terms Since it is very desirable to have the solution as short as possible astronomers have devoted much attention to this equation, and several hundred methods of solving it have been discovered

94 Geometrical Derivation of Kepler's Equation Construct the ellipse in which the body moves, and its auxiliary circle AQB The angle AFP equals the true anomaly, v, the angle ACQ

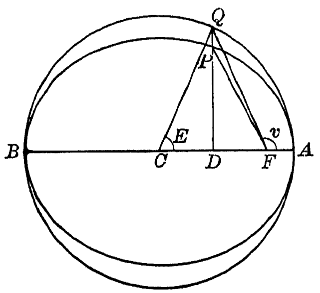

Fig 27

will be defined as the eccentric anomaly, E, and it will be shown that the relation between M and E is given by Kepler's equation From the law of areas and the properties of the auxiliary circle, it follows that

$$\frac{M}{2\pi} = \frac{\text{area } AFP}{\text{area ellipse}} = \frac{\text{area } AFQ}{\text{area circle}}$$

$$\text{Area } AFQ = \text{area } ACQ - \text{area } FCQ = \frac{a^2 E}{2} - \frac{a}{2}ae\sin E$$

Therefore

$$\frac{M}{2\pi} = \frac{a^2}{2} \frac{(E - e \sin E)}{\pi a^2},$$

or,

$$\begin{cases} M = E - e \sin E, \\ FP = r = \dfrac{a(1-e^2)}{1 + e\cos v} = \sqrt{\overline{PD}^2 + \overline{FD}^2} = a(1 - e\cos E), \end{cases}$$

which is the definition of the eccentric anomaly given in (41)

95 Solution of Kepler's Equation It will be shown first that Kepler's equation always has one, and only one, real solution for every value of M and for every e such that $0 < e < 1$ Write the equation in the form

$$\phi(E) \equiv E - e \sin E - M = 0$$

Suppose M has some given value between $n\pi$ and $(n+1)\pi$, where n is any integer, then there is but one real value of E satisfying this equation, and it lies between $n\pi$ and $(n+1)\pi$ For, the function $\phi(E)$ for $E = n\pi$ is

$$\phi(n\pi) = n\pi - M < 0$$

And $\phi(E)$ for $E = (n+1)\pi$ is

$$\phi[(n+1)\pi] = (n+1)\pi - M > 0$$

Consequently there is an odd number of real solutions for E which lie between $n\pi$ and $(n+1)\pi$ But the derivative

$$\phi'(E) \equiv 1 - e \cos E$$

is always positive, therefore $\phi(E)$ increases continually with E and takes the value zero but once Q E D

A convenient method of practically solving the equation is by means of an expansion due to Lagrange Suppose z is defined as a function of w by the equation

$$z = w + \alpha\phi(z), \tag{43}$$

where α is a parameter Lagrange has shown that any function of z can be expressed in a power series in α, which converges for sufficiently small values of α, of the form*

$$\text{(44)}\quad \begin{cases} F(z) = F(w) + \dfrac{\alpha}{1}\ \phi(w) F'(w) + \dfrac{\alpha^2}{1\ 2} \dfrac{\partial}{\partial w}[\{\phi(w)\}^2 F'(w)] \\ \qquad + \quad + \dfrac{\alpha^{n+1}}{(n+1)!} \dfrac{\partial^n}{\partial w^n}[\{\phi(w)\}^{n+1} F'(w)] + \end{cases}$$

* Williamson's *Diff Calc* p 151

This expansion may be applied to the solution of Kepler's equation by writing it

$$E = M + e \sin E,$$

which is of the same form as (43) The expansion of E in a series in e may be taken from (44) by putting $F(z) = E$, $\phi(z) = \sin E$, $w = M$, and $a = e$ The result is

$$E = M + \frac{e}{1} \sin M + \frac{e^2}{1 \quad 2} \sin 2M + \qquad (45)$$

All the terms on the right except the first are expressed in radians and must be reduced to degrees by multiplying each of them by the number of degrees in a radian The higher terms are considerably more complicated than those written, and the work of computing them increases very rapidly In the planetary and satellite orbits the eccentricity is very small, and the series (45) converges with great rapidity, the first three terms giving quite an accurate value of E

96 Differential Corrections A method is about to be explained in one of its simplest applications, which is of great importance in many astronomical problems Suppose an approximate value of E is determined by the first three terms of (45) Call it E_0, that is,

$$E_0 = M + e \sin M + \frac{e^2}{2} \sin 2M$$

It is required to find the correct value of E

Kepler's equation gives

$$M_0 = E_0 - e \sin E_0$$

For a particular value of M, viz M_0, the corresponding value of E is known, viz E_0 It is required to find the value of E corresponding to M, which differs only a little from M_0 M is a function of E and may be written

$$M = f(E),$$

or,

$$M = M_0 + \Delta M_0 = f(E_0 + \Delta E_0)$$

Expanding by Taylor's formula, this becomes

$$M = M_0 + \Delta M_0 = f(E_0) + f'(E_0) \Delta E_0 +$$

By the definitions of the quantities, $M_0 = f(E_0)$, therefore this equation may be written

$$M - M_0 = f'(E_0) \Delta E_0 + \qquad (46)$$

From Kepler's equation it is found that

$$f'(E_0) = 1 - e \cos E_0$$

Since ΔE_0 is very small the squares and higher powers may be neglected*, when equation (46) gives the correction to be applied to E_0,

$$\Delta E_0 = \frac{M - M_0}{1 - e \cos E_0} \tag{47}$$

With the more nearly correct value of E, $E_1 = E_0 + \Delta E_0$, M_1 may be computed from Kepler's equation, and a second correction will be

$$\Delta E_1 = \frac{M - M_1}{1 - e \cos E_1}$$

This process may be repeated until the value of E is found as near as may be desired† In the planetary orbits two applications of the formulas will nearly always give results which are sufficiently accurate, and usually one correction will suffice

97 Graphical Solution of Kepler's Equation When the eccentricity is more than 0 2 the method of solving Kepler's equation given above is laborious because the first approximation will be very inexact These high eccentricities occur in binary star and comet orbits, and are sometimes even so great as 0 9 In the case of binary star orbits it is usually sufficient to have a solution to within a tenth of one degree In this work a rapid graphical method is of great practical value

Consider Kepler's equation

$$E - e \sin E - M = 0,$$

where M is given and E is required Take a rectangular system of axes and construct the curve and straight line whose equations are

$$\begin{cases} y = \sin E, \\ y = \dfrac{1}{e}(E - M) \end{cases}$$

The abscissa of their point of intersection is the value of E satisfying the equation‡, for, eliminating y, Kepler's equation results The first curve is the familiar sine curve which may be constructed once for all, the second is a straight line making with the E-axis an angle whose tangent is $\frac{1}{e}$ Instead of drawing the straight line a straight-edge may

* If the higher terms in ΔE_0 were not neglected ΔE_0 could be expressed as a power series in $M - M_0$, of which the first term would be the right member of (47)

† For the proof of the convergence of a similar, but somewhat more laborious process, see Appell's *Mécanique*, vol I p 391

‡ Due to J J Waterson, *Monthly Notices*, 1849–50, p 169

be laid down making the proper slope with the axis To facilitate the determination of its position construct a line with the degrees marked on it at an altitude of 100*, then place the bottom of the straight-edge at M and the top at $M + 100e$, and it follows that it will have the

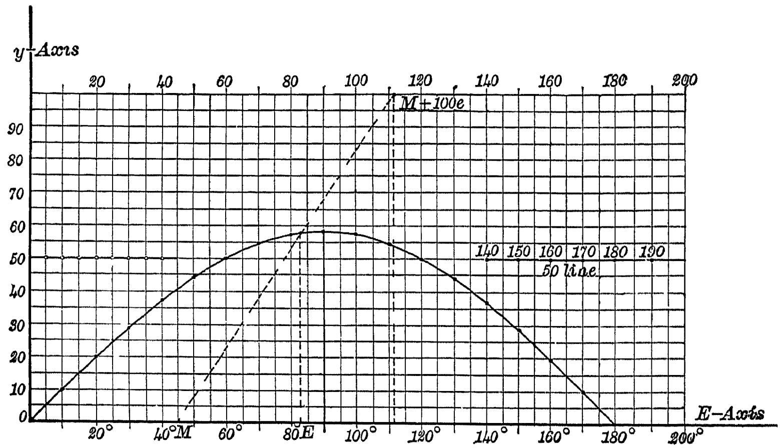

Fig 28

proper slope If M is so near 180 that the straight-edge runs off from the diagram, the top may be placed at $M + 50e$ on the 50-line As M becomes very near 180 the mean and eccentric anomalies become very nearly equal, exactly coinciding at $M = 180°$

98 Recapitulation of Formulas The equations for the computation of the polar coordinates, when the time is given, will now be given in the order in which they are used

$$
(48)\qquad \left\{
\begin{aligned}
n &= \frac{k\sqrt{1+m}}{a^{\frac{3}{2}}}, \\
M &= n(t-T), \\
E_0 &= M + e\sin M + \frac{e^2}{2}\sin 2M, \\
M_0 &= E_0 - e\sin E_0, \\
\Delta E_0 &= \frac{M - M_0}{1 - e\cos E_0}, \\
E_1 &= E_0 + \Delta E_0, \\
r &= a(1 - e\cos E) = \frac{a(1-e^2)}{1 + e\cos v},
\end{aligned}
\right.
$$

* This device is due to Professor C A Young

whence

$$
(49)\quad \begin{cases}
\cos v = \dfrac{\cos E - e}{1 - e\cos E}, \\[2ex]
\sin v = \dfrac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \\[2ex]
1 + \cos v = \dfrac{(1-e)(1+\cos E)}{1 - e\cos E}, \\[2ex]
1 - \cos v = \dfrac{(1+e)(1-\cos E)}{1 - e\cos E}
\end{cases}
$$

The square root of the quotient of the last two equations gives a very convenient formula for the computation of v, viz

$$
(50)\qquad \sqrt{\frac{1-\cos v}{1+\cos v}} = \tan\frac{v}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}
$$

The last equation of (48) and equation (50) give the polar coordinates when E is known

99 Developments in Series The equations which have been given are sufficient to enable one to compute the polar, and consequently the rectangular, coordinates at any epoch, yet, in some kinds of investigations, as in the theory of perturbations, it is necessary to have the developments of not only E, but also the polar coordinates, carried so far that the functions are represented by the series with the desired degree of accuracy The most important of these developments will be given in this article They are most conveniently found by means of Bessel's functions, but, as this method of derivation would require a somewhat lengthy digression, they will be constructed from the Lagrangian expansion (44)

Carrying out the expansion indicated in Art 95, the value of E to terms of the sixth order in e is found to be

$$
(51)\quad \begin{cases}
E = M + e\sin M + \dfrac{e^2}{2}\sin 2M \\[2ex]
\quad + \dfrac{e^3}{3!\,2^2}(3^2\sin 3M - 3\sin M) \\[2ex]
\quad + \dfrac{e^4}{4!\,2^3}(4^3\sin 4M - 4\cdot 2^3\sin 2M) \\[2ex]
\quad + \dfrac{e^5}{5!\,2^4}(5^4\sin 5M - 5\cdot 3^4\sin 3M + 10\sin M) \\[2ex]
\quad + \dfrac{e^6}{6!\,2^5}(6^5\sin 6M - 6\cdot 4^5\sin 4M + 15\cdot 2^5\sin 2M) \\[2ex]
\quad + \cdots
\end{cases}
$$

The radius vector is expressed in terms of E by (41) To have it expressed in terms of M it is only necessary to put $F(z) = \cos E$, $\phi(z) = \sin E$, $w = M$, and $\alpha = e$ in (44) Making these substitutions, and reducing all powers of the trigonometrical functions to the cosines of multiples of M, it is found that

$$(52)\quad \left\{ \begin{aligned} \frac{r}{a} = 1 - e \cos M - \frac{e}{2}(\cos 2M - 1) - \frac{e^3}{2!\,2^2}(3 \cos 3M - 3 \cos M) \\ - \frac{e^4}{3!\,2^3}(4^2 \cos 4M - 4 \quad 2 \quad \cos 2M) \\ - \frac{e^5}{4!\,2^4}(5^3 \cos 5M - 5 \quad 3^3 \cos 3M + 10 \cos M) \\ - \frac{e^6}{6!\,2^5}(6^4 \cos 6M - 6 \quad 4^4 \cos 4M + 15 \quad 2^4 \cos 2M) \\ - \end{aligned} \right.$$

Referring to equations (49) it will be seen that v cannot be expressed in terms of E in a simple manner, consequently it is a complicated matter to express it in terms of M Instead of expressing v directly in terms of E it is simpler to express $\frac{dv}{dt}$ in terms of E, and through this relation in terms of M, and then to integrate the final expression

From the integral of areas it follows that

$$(53)\qquad dv = \frac{k}{r^2}\sqrt{M_{1\ 2}\,a(1-e)}\,dt = \sqrt{1-e}\ \frac{a^2}{r}\,n\,dt = \sqrt{1-e^2}\,\frac{a^2}{r^2}\,dM$$

By equation (41),

$$\frac{a^2}{r^2} = (1 - e \cos E)^{-2}$$

Consequently, in applying (44), it is necessary to let

$$F(z) = (1 - e \cos E)^{-2}, \quad \phi(z) = \sin E, \quad w = M, \text{ and } \alpha = e$$

Making these substitutions, and reducing all powers of the trigonometrical functions to the cosines of the multiples of M, it is found that

$$\begin{aligned} \frac{a^2}{r^2} = 1 + 2e \cos M + \frac{e}{2}(5 \cos 2M + 1) + \frac{e^3}{4}(13 \cos 3M + 3 \cos M) \\ + \frac{e^4}{24}(103 \cos 4M + 8 \cos 2M + 9) \\ + \frac{e^5}{192}(1097 \cos 5M - 75 \cos 3M + 130 \cos M) \\ + \frac{e^6}{480}(3669 \cos 6M - 774 \cos 4M + 315 \cos 2M + 150) \\ + \end{aligned}$$

Multiplying by the expansion of $\sqrt{1-e^2}$, substituting in (53), and integrating, it is found that

$$
(54)\quad \begin{cases}
v = M + 2e\sin M + \frac{5}{4}e^2\sin 2M + \frac{e^3}{12}(13\sin 3M - 3\sin M) \\
\quad + \frac{e^4}{96}(103\sin 4M - 44\sin 2M) \\
\quad + \frac{e^5}{960}(1097\sin 5M - 645\sin 3M + 50\sin M) \\
\quad + \frac{e^6}{960}(1223\sin 6M - 902\sin 4M + 85\sin 2M) \\
\quad + \cdots
\end{cases}
$$

When e is small, as in the planetary orbits, these series are very rapidly convergent. If e exceeds 0.6627 ... they diverge, as Laplace first showed, for some values of M. This value of e is exceeded in the solar system only in the case of some of the comets' orbits, but developments of this sort are not employed in computing the perturbations of the comets.

100. Position in Hyperbolic Orbits. The polar equation of the conic and the integral of areas are respectively

$$
\begin{cases}
r = \dfrac{a(\epsilon^2-1)}{1+\epsilon\cos v}, \\[2ex]
r^2\dfrac{dv}{dt} = k\sqrt{M_{1,2}\,a(\epsilon^2-1)},
\end{cases}
$$

where ϵ represents the eccentricity. In the polar equation of the hyperbola v can vary only from $-\pi+\cos^{-1}\left(\frac{1}{\epsilon}\right)$ to $+\pi-\cos^{-1}\left(\frac{1}{\epsilon}\right)$.

From the first equation above

$$
\begin{cases}
\cos v = \dfrac{a(\epsilon^2-1)}{\epsilon r} - \dfrac{1}{\epsilon}, \\[2ex]
dv = \dfrac{a\sqrt{\epsilon^2-1}}{r}\,\dfrac{dr}{\sqrt{(a+r)^2-a^2\epsilon^2}}.
\end{cases}
$$

Then the second becomes

$$\nu dt = \frac{r}{a}\,\frac{dr}{\sqrt{(a+r)^2-a^2\epsilon^2}},$$

where $\nu = \dfrac{k\sqrt{M_{1,2}}}{a^{\frac{3}{2}}}$.

This equation can be integrated at once in terms of logarithmic functions, but it is preferable to introduce first an auxiliary quantity, F, corresponding to the eccentric anomaly in elliptic orbits Let

$$(55) \qquad a + r = \frac{a\epsilon}{2}(e^F + e^{-F}) = a\epsilon \cosh F,$$

then
$$\nu dt = \left\{-1 + \frac{\epsilon}{2}(e^F + e^{-F})\right\} dF = \{-1 + \epsilon \cosh F\}\, dF$$

The integral of this equation is

$$(56) \qquad M = \nu(t - T) = -F + \frac{\epsilon}{2}(e^F - e^{-F}) = -F + \epsilon \sinh F,$$

which gives t when F is known The inverse problem of finding F when $\nu(t - T)$ is given is one of more difficulty The most expeditious method would be, in general, to find an approximate value of F by some graphical process, and then a more exact value by differential corrections The value of F satisfying (56) is the abscissa of the point of intersection of the line

$$y = \frac{1}{\epsilon}(F + M),$$

and the curve
$$y = \frac{e^F - e^{-F}}{2}$$

The differential corrections would be computed in a manner analogous to that developed in the case of the elliptic orbits

From (55) and the polar equation of the hyperbola, it follows that

$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos v} = a\left\{-1 + \frac{\epsilon}{2}(e^F + e^{-F})\right\},$$

and from this equation,

$$\tan\frac{v}{2} = \sqrt{\frac{\epsilon + 1}{\epsilon - 1}}\, \frac{\sqrt{-1 + \frac{1}{2}(e^F + e^{-F})}}{\sqrt{1 + \frac{1}{2}(e^F + e^{-F})}} = \sqrt{\frac{\epsilon + 1}{\epsilon - 1}} \tanh\frac{F}{2},$$

which is a convenient formula for computing v when F has been found

101 Position in Elliptic Orbits when the Eccentricity is Nearly Equal to Unity The analytical solutions heretofore given have depended upon expansions in powers of e If e is large, as in the case of some of the periodic comets' orbits, the convergence ceases or is so slow that the methods become impracticable The graphical process, however, avoids this difficulty

In order to obtain a workable analytical solution, the developments may be made in powers of $\frac{1-e}{1+e}$ The start is made again from the equation of areas and the polar equation of the ellipse

$$\begin{cases} r^2 \dfrac{dv}{dt} = na^2\sqrt{1-e^2}, \\ r = \dfrac{a(1-e^2)}{1+e\cos v} \end{cases}$$

Let

$$\begin{cases} w = \tan \dfrac{v}{2}, \\ \lambda = \dfrac{1-e}{1+e}, \end{cases}$$

then the equation of areas becomes

$$\frac{n\sqrt{1+e}\,dt}{2(1-e)^{\frac{3}{2}}} = \frac{(1+w^2)}{(1+\lambda w)^2}\,dw$$

λ is very small and the right member may be developed into a rapidly converging series in λ for all values of v not too near 180° Since the periodic comets are always invisible when near aphelion, there will seldom be occasion to consider the solution in this region Expanding the right member and integrating, the result is

$$(57)\qquad \frac{n\sqrt{1+e}\,(t-T)}{2(1-e)^{\frac{3}{2}}} = w + \frac{w^3}{3} - 2\lambda\left(\frac{w^3}{3} + \frac{w^5}{5}\right) + 3\lambda^2\left(\frac{w^5}{5} + \frac{w^7}{7}\right)$$
$$- 4\lambda^3\left(\frac{w^7}{7} + \frac{w^9}{9}\right) +$$

When the orbit is a parabola $e = 1$ and $\lambda = 0$, and this equation reduces to (32), which is a cubic in w Since the perihelion distance in an ellipse is $q = a(1-e)$ and $n = \frac{k}{a^{\frac{3}{2}}}$, it follows that

$$\frac{n\sqrt{1+e}}{2(1-e)^{\frac{3}{2}}} = \frac{k\sqrt{1+e}}{2q^{\frac{3}{2}}}$$

It is desired to find the value of w for any value of t If the eccentricity should become equal to unity, the left member keeping the same value, equation (57) would have the form

$$(58)\qquad \frac{k\sqrt{1+e}\,(t-T)}{2q^{\frac{3}{2}}} = W + \tfrac{1}{3}W^3,$$

where W would be the tangent of half the true anomaly in the resulting parabolic orbit From this equation W may be determined

by means of Barker's tables, or from equations (33) Suppose W has been found, then w may be expressed as a series in λ of which the coefficients are functions of W For, assume the development

$$w = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \alpha_3\lambda^3 + \quad , \tag{59}$$

substitute it in the right member of (57), which is equal to the right member of (58) The result of the substitution is

$$\begin{aligned} W + \frac{W^3}{3} = \alpha_0 + \frac{\alpha_0^3}{3} &+ (\alpha_1 + \alpha_0^2\alpha_1 - \tfrac{2}{3}\alpha_0^3 - \tfrac{2}{5}\alpha_0^5)\,\lambda \\ &+ (\alpha_2 + \alpha_0^2\alpha_2 + \alpha_0\alpha_1^2 - 2\alpha_0\ \alpha_1 - 2\alpha_0^4\alpha_1 + \tfrac{3}{5}\alpha_0^5 + \tfrac{3}{7}\alpha_0^7)\,\lambda^2 \\ &+ \Big(\alpha_3 + \alpha_0^2\alpha_3 + \frac{\alpha_1^3}{3} - 2\alpha_0^2\alpha_2 - 2\alpha_0^4\alpha_2 - 2\alpha_0\alpha_1^2 - 4\alpha_0^3\alpha_1^2 \\ &\qquad + 3\alpha_0^4\alpha_1 + 3\alpha_0^6\alpha_1 - \tfrac{4}{7}\alpha_0^7 - \tfrac{4}{9}\alpha_0^9\Big)\,\lambda^3 \\ &+ \end{aligned}$$

Since this equation is an identity in λ the coefficients of corresponding powers of λ are equal Hence

$$\left\{\begin{aligned} &\alpha_0 + \frac{\alpha_0^3}{3} = W + \frac{W^3}{3}, \\ &\alpha_1(1+\alpha_0^2) = \tfrac{2}{3}\alpha_0^3 + \tfrac{2}{5}\alpha_0^5, \\ &\alpha_2(1+\alpha_0^2) = -\alpha_0\alpha_1^2 + 2\alpha_0^2\alpha_1 + 2\alpha_0^4\alpha_1 - \tfrac{3}{5}\alpha_0^5 - \tfrac{3}{7}\alpha_0^7, \\ &\alpha_3(1+\alpha_0^2) = -\frac{\alpha_1^3}{3} + 2\alpha_0^2\alpha_2 + 2\alpha_0^4\alpha\ + 2\alpha_0\alpha_1^2 + 4\alpha_0^3\alpha_1^2 - 3\alpha_0^4\alpha_1 - 3\alpha_0^6\alpha_1 \\ &\qquad\qquad + \tfrac{4}{7}\alpha_0^7 + \tfrac{4}{9}\alpha_0^9, \end{aligned}\right.$$

There are three solutions for α_0, only one of which is real Taking the real root of the first equation, it is found that

$$\left\{\begin{aligned} &\alpha_0 = W, \\ &\alpha_1 = \frac{2\left(\dfrac{W^3}{3} + \dfrac{W^5}{5}\right)}{1+W^2}, \\ &\alpha_2 = \frac{\frac{11}{15}W^5 + \frac{439}{315}W^7 + \frac{33}{35}W^9 + \frac{37}{175}W^{11}}{(1+W^2)^3}, \\ &\alpha_3 = \frac{\frac{292}{315}W^7 + \frac{7928}{2835}W^9 + \frac{10328}{2835}W^{11} + \frac{432}{175}W^{13} + \frac{6692}{7875}W^{15} + \frac{184}{1575}W^{17}}{(1+W^2)^5}, \end{aligned}\right.$$

Substituting the values of these coefficients in (59) the tangent of one-half the true anomaly is determined The first term gives that which would come from a parabolic orbit, the remaining terms vanishing as $e=1$ In the series (54) the first term in the right member

would be the true anomaly if the orbit were a circle, the higher terms being the corrections to circular motion In the series (59) the first term in the right member would give the tangent of one-half the true anomaly if the orbit were a parabola, the higher terms being the corrections to parabolic motion

It should be added that these equations apply equally to hyperbolic orbits in which the eccentricity is near unity

XV PROBLEMS

1 Show how the cubic equation (32) may be solved approximately for $\tan\frac{v}{2}$ with great rapidity by a graphical construction

2 Develop the equations for differential corrections to the approximate values found by the graphical method. Apply to a particular problem and verify the result

3 If $e=0\ 2$ and $M=214^\circ$, find E_0, M_0, E_1, M_1, E_2, and M_2

Ans $E_0=208^\circ 39' 16''\ 6$, $M_0=214^\circ 8\ 58''\ 6$, $E_1=208^\circ 31' 38''\ 4$,
$M_1=213\ 59' 59''\ 8$, $E_2=208^\circ 31' 38''\ 6$, $M_2=214^\circ 00' 00''$

4 Show from the curves employed in solving Kepler's equation that the solution is unique for all values of e and M

5 In (50) the quadrant is not determined by the equation, show that corresponding values of v and E always lie in the same quadrant

6 Express the rectangular coordinates $x=r\cos v$, $y=r\sin v$ in terms of the eccentric anomaly, and then by means of the Lagrange expansion formula in terms of M

$$
Ans\ \begin{cases} \dfrac{x}{a}=\cos M+\dfrac{e}{2}(\cos 2M-3)+\dfrac{e^2}{2!\,2^2}(3\cos 3M-3\cos M) \\ \qquad +\dfrac{e^3}{3!\,2^3}(4^2\cos 4M-4\ \ 2^2\cos 2M)+ \\ \dfrac{y}{a}=\sin M+\dfrac{e}{2}\sin 2M+\dfrac{e^2}{3!\,2^2}(3^2\sin 3M-15\sin M) \\ \qquad +\dfrac{e^3}{4!\,2^3}(4^3\sin 4M-10\ \ 2^3\sin 2M)+ \end{cases}
$$

7 Give a geometrical interpretation of F (Art 100) corresponding to that of E in an elliptic orbit

8 Express v as a power series in e by a method analogous to that used in Art 101

102 The Heliocentric Position in the Ecliptic System Methods have been given for finding the positions in the orbits in the various cases which arise The formulas must now be derived for determining the position referred to different systems of axes The origin will first be kept fixed at the body with respect to which the motion of the second is given Since most of the applications are in the solar system where the origin is at the center of the sun, the coordinates will be called *heliocentric*

Positions of bodies in the solar system are usually referred to one of two systems of coordinates, the *ecliptic* system, or the *equatorial* system The fundamental plane in the ecliptic system is the plane of the earth's orbit, in the equatorial system it is the plane of the earth's equator The zero point of the fundamental circles in both systems is the *vernal equinox*, or the point at which the ecliptic cuts the equator from south to north, and is denoted by ♈ The polar coordinates in the ecliptic system are called *longitude* and *latitude*, and in the equatorial, *right ascension* and *declination* When the origin is at the sun Roman letters are used to represent the coordinates, and when at the earth, Greek Thus

	Origin at sun	*Origin at earth*	
longitude	l	λ	
latitude	b	β	+ if north, − if south
right ascension	a	α	
declination	d	δ	+ if north, − if south
distance	r	ρ	

In practice a and d are very seldom used Absolute positions of fundamental stars are given in the equatorial system, and the observed positions of comets are determined by comparison with them But in determining comets' orbits it is more convenient to use the ecliptic system, so it is necessary to transform the equations from one system to the other

The *ascending node* is the projection on the ecliptic, from the sun, of the place at which the body crosses the plane of the ecliptic from south to north It is measured from a fixed point in the ecliptic, the vernal equinox, and is denoted by ☊ The projection of the point where the body crosses the plane of the ecliptic from north to south is called the *descending node*, and is denoted by ☋

The *inclination* is the angle between the plane of the orbit and the plane of the ecliptic, and is denoted by i It has been the custom of some writers to take the inclination always less than 90°, and to define

the direction of motion as *direct* or *retrograde*, according as it is the same as that of the earth or the opposite. Another method that has been used is to consider all motion direct and the inclination as varying from 0° to 180°. The latter method avoids the use of double signs in the formulas and will be adopted here. The node and inclination define the position of the plane of the orbit in space.

The distance from the ascending node to the perihelion point counted in the direction of the motion of the body in its orbit is ω, and defines the orientation of the orbit in its plane. The *longitude of the perihelion* is denoted by π and is given by the equation

$$\pi = \Omega + \omega$$

The problem of relative motion of two bodies was of the sixth order (Art 85), and in the integration six arbitrary constants were introduced. There are six elements, therefore, which are independent functions of these constants. They are

a = major semi-axis, which defines the size of the orbit and the period of revolution.

e = the eccentricity, which defines the shape of the orbit.

Ω = longitude of ascending node, and

i = inclination to plane of the ecliptic, which together define the position of the plane of the orbit.

ω = longitude of the perihelion point from the node, or π = longitude of the perihelion, either defining the orientation of the orbit in its plane.

T = time of perihelion passage, defining, with the other elements, the position of the body in its orbit at any time.

The polar coordinates have been computed; hence the rectangular coordinates with the positive end of the x-axis directed to the perihelion point and the y-axis in the plane of the orbit are given by the equations

$$(60) \qquad \begin{cases} x_0 = r \cos v, \\ y_0 = r \sin v, \\ z_0 = 0 \end{cases}$$

If the x-axis be rotated backward to the line of nodes the coordinates will be in the new system

$$(61) \qquad \begin{cases} x = r \cos (v + \omega) = r \cos (v + \pi - \Omega), \\ y = r \sin (v + \omega) = r \sin (v + \pi - \Omega), \\ z = 0 \end{cases}$$

The longitude of the body in its orbit counted from the ascending node is called the *argument of the latitude* and is denoted by u It is given by the equation

$$u = v + \omega,$$

hence u is known when v has been found

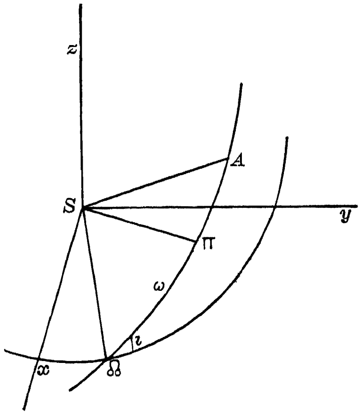

Fig 29

Let S represent the sun and Sxy the plane of the ecliptic, $S☊A$, the plane of the orbit, ☊, the ascending node, Π, the perihelion point, A, the projection of the position of the body, and angle $\Pi SA = v$ Then $☊A = \omega + v = u$

Let the position of the body be referred now to a rectangular system of axes with the origin at the sun, the x-axis in the line of the nodes, and the y-axis in the plane of the ecliptic Then equations (61) become

$$(62) \qquad \begin{cases} x' = r\cos(v+\omega) = r\cos u, \\ y\ = r\sin(v+\omega)\cos i = r\sin u\cos i, \\ z' = r\sin(v+\omega)\sin i = r\sin u\sin i \end{cases}$$

But, in terms of the heliocentric latitude and longitude,

$$(63) \qquad \begin{cases} x' = r\cos b\cos(l - ☊), \\ y' = r\cos b\sin(l - ☊), \\ z' = r\sin b \end{cases}$$

Therefore, comparing (62) and (63),

$$(64) \qquad \begin{cases} \cos b\cos(l - ☊) = \cos u, \\ \cos b\sin(l - ☊) = \sin u\cos i, \\ \sin b \qquad\qquad\quad = \sin u\sin i, \end{cases}$$

whence

$$(65)\qquad \begin{cases} \tan(l - \Omega) = \tan u \cos i, \\ \tan b \qquad\quad = \tan i \sin(l - \Omega) \end{cases}$$

Since $\cos b$ is always positive equations (64) and (65) determine the heliocentric longitude and latitude, l and b, uniquely when Ω, i, and u are known

103 Transfer of the Origin to the Earth Let Ξ, H, Z be the geocentric coordinates of the center of the sun referred to a system of axes with the x-axis directed to the vernal equinox, and the y-axis in the plane of the ecliptic Let P, Λ, and B* represent the geocentric distance, longitude, and latitude of the sun respectively These quantities are given in the *Nautical Almanac* for every day in the year The rectangular coordinates are expressed in terms of them by

$$(66)\qquad \begin{cases} \Xi = \mathrm{P}\cos \mathrm{B}\cos \Lambda, \\ \mathrm{H} = \mathrm{P}\cos \mathrm{B}\sin \Lambda, \\ \mathrm{Z} = \mathrm{P}\sin \mathrm{B} \end{cases}$$

B is generally less than a second of arc, and unless great accuracy is required these equations may be replaced by

$$\begin{cases} \Xi = \mathrm{P}\cos \Lambda, \\ \mathrm{H} = \mathrm{P}\sin \Lambda, \\ \mathrm{Z} = 0 \end{cases}$$

Let ξ'', η'', and ζ'' be the geocentric, and x'', y'', and z'' the heliocentric, coordinates of the body with the x-axis directed to the vernal equinox and the y-axis in the plane of the ecliptic Therefore

$$\begin{cases} \xi' = x'' + \Xi, \\ \eta'' = y'' + \mathrm{H}, \\ \zeta'' = z'' + \mathrm{Z} \end{cases}$$

These equations are, in polar coordinates,

$$\begin{cases} \rho\cos\beta\cos\lambda = r\cos b\cos l + \mathrm{P}\cos \mathrm{B}\cos\Lambda, \\ \rho\cos\beta\sin\lambda = r\cos b\sin l + \mathrm{P}\cos \mathrm{B}\sin\Lambda, \\ \rho\sin\beta \qquad\quad = r\sin b \qquad\quad + \mathrm{P}\sin \mathrm{B} \end{cases}$$

From these equations λ and β may be found, but this system may be transformed into one which is more convenient by multiplying the first equation by $\cos\Lambda$, the second by $\sin\Lambda$, and adding the products, and

* P, Λ, B = capital ρ, λ, β

then multiplying the first by $-\sin\Lambda$ and the second by $\cos\Lambda$, and adding the products The results are

$$(67)\quad \begin{cases} \rho\cos\beta\cos(\lambda-\Lambda) = r\cos b\cos(l-\Lambda) + \mathrm{P}\cos\mathrm{B}, \\ \rho\cos\beta\sin(\lambda-\Lambda) = r\cos b\sin(l-\Lambda), \\ \rho\sin\beta \qquad\qquad\quad = r\sin b \qquad\qquad\quad + \mathrm{P}\sin\mathrm{B} \end{cases}$$

These equations give the geocentric distance, longitude, and latitude, ρ, λ, and β

104 Transformation to Geocentric Equatorial Coordinates Let ϵ represent the inclination of the plane of the ecliptic to the plane of the equator Let ξ'', η'', and ζ'' be the geocentric coordinates of the body referred to the ecliptic system with the x-axis directed to the vernal equinox Then the equatorial system may be obtained by rotating the ecliptic system around the x-axis in the negative direction through the angle ϵ, the relations between the coordinates in the two systems being

$$\begin{cases} \xi''' = \xi'', \\ \eta''' = \eta''\cos\epsilon - \zeta''\sin\epsilon, \\ \zeta''' = \eta''\sin\epsilon + \zeta''\cos\epsilon, \end{cases}$$

or, in polar coordinates,

$$(68)\quad \begin{cases} \cos\delta\cos\alpha = \cos\beta\cos\lambda, \\ \cos\delta\sin\alpha = \cos\beta\sin\lambda\cos\epsilon - \sin\beta\sin\epsilon, \\ \sin\delta \qquad\quad = \cos\beta\sin\lambda\sin\epsilon + \sin\beta\cos\epsilon \end{cases}$$

In order to solve these equations conveniently for δ and α the auxiliaries n and N will be introduced by the equations

$$(69)\quad \begin{cases} n\sin N = \sin\beta, \\ n\cos N = \cos\beta\sin\lambda, \end{cases}$$

in which n is to be taken always positive Then equations (68) become

$$\begin{cases} \cos\delta\cos\alpha = \cos\beta\cos\lambda, \\ \cos\delta\sin\alpha = n\cos(N+\epsilon), \\ \sin\delta \qquad\quad = n\sin(N+\epsilon), \end{cases}$$

whence

$$(70)\quad \begin{cases} n\sin N = \sin\beta, \\ n\cos N = \cos\beta\sin\lambda, \\ \tan\alpha = \dfrac{\cos(N+\epsilon)\tan\lambda}{\cos N}, \\ \tan\delta = \tan(N+\epsilon)\sin\alpha \end{cases}$$

These equations, together with the first of (68) which is used in determining the quadrant in which α lies, give α and δ without ambiguity when λ and β are known

If α and δ are given and λ and β are required, the equations from which they may be computed are found by interchanging α and δ with λ and β, and changing ϵ to $-\epsilon$ in (70) They are

$$(71)\qquad \begin{cases} m \sin M^* = \sin \delta, \\ m \cos M \ = \cos \delta \sin \alpha, \\ \tan \lambda = \dfrac{\cos (M-\epsilon) \tan \alpha}{\cos M}, \\ \tan \beta = \tan (M-\epsilon) \sin \lambda \end{cases}$$

105 Direct Computation of the Geocentric Equatorial Coordinates The geocentric equatorial coordinates, α and δ, may be found directly from the elements, i and ☊, and the argument of the latitude u, without first finding the ecliptic coordinates, λ and β

In a system of axes with the x-axis directed to the node and the y-axis in the plane of the ecliptic, the equations for the heliocentric coordinates are

$$\begin{cases} x' = r \cos u, \\ y' = r \sin u \cos i, \\ z' = r \sin u \sin i \end{cases}$$

If the system be rotated around the z-axis until the x-axis is directed to the vernal equinox, the new coordinates are

$$\begin{cases} x'' = x' \cos ☊ - y' \sin ☊, \\ y'' = x' \sin ☊ + y' \cos ☊, \\ z'' = z', \end{cases}$$

or,

$$(72)\qquad \begin{cases} x'' = r (\cos u \cos ☊ - \sin u \cos i \sin ☊), \\ y'' = r (\cos u \sin ☊ + \sin u \cos i \cos ☊), \\ z'' = r \sin u \sin i \end{cases}$$

If the system be rotated now around the x-axis through the angle $-\epsilon$, the new coordinates will be

$$\begin{cases} x''' = x'', \\ y''' = y'' \cos \epsilon - z'' \sin \epsilon, \\ z''' = y'' \sin \epsilon + z'' \cos \epsilon, \end{cases}$$

* m and M are new auxiliaries, not being related to any of the quantities which they previously have represented

or, in polar coordinates,

$$(73)\quad \begin{cases} x''' = r\{\cos u \cos \Omega - \sin u \cos i \sin \Omega\}, \\ y''' = r\{(\cos u \sin \Omega + \sin u \cos i \cos \Omega) \cos \epsilon - \sin u \sin i \sin \epsilon\}, \\ z''' = r\{(\cos u \sin \Omega + \sin u \cos i \cos \Omega) \sin \epsilon + \sin u \sin i \cos \epsilon\} \end{cases}$$

In order to facilitate the computation Gauss introduced the new auxiliaries A, a, B, b, C, and c by the equations

$$(74)\quad \begin{cases} \sin a \sin A = \cos \Omega, & \\ \sin a \cos A = -\sin \Omega \cos i, & \sin a > 0, \\ \sin b \sin B = \sin \Omega \cos \epsilon, & \sin b > 0, \\ \sin b \cos B = \cos \Omega \cos i \cos \epsilon - \sin i \sin \epsilon, & \\ \sin c \sin C = \sin \Omega \sin \epsilon, & \sin c > 0, \\ \sin c \cos C = \cos \Omega \cos i \sin \epsilon + \sin i \cos \epsilon & \end{cases}$$

These constants depend upon the elements alone, so they need be computed but once for a given orbit They are of particular advantage when the coordinates are to be computed for a large number of epochs, as in constructing an ephemeris When these constants are substituted in (73) these equations for the heliocentric coordinates take the simple form

$$(75)\quad \begin{cases} x''' = r \sin a \sin (A + u), \\ y''' = r \sin b \sin (B + u), \\ z''' = r \sin c \sin (C + u), \end{cases}$$

from which x''', y''', and z''' may be found

Then finally, the geocentric equatorial coordinates are defined by

$$(76)\quad \begin{cases} \rho \cos \delta \cos \alpha = x''' + X', \\ \rho \cos \delta \sin \alpha = y''' + Y', \\ \rho \sin \delta \qquad\quad = z''' + Z', \end{cases}$$

where X', Y', and Z' are the rectangular geocentric coordinates of the sun referred to the equatorial system They are given in the *Nautical Almanac* for every day in the year, and, therefore, these equations define ρ, α, and δ

This completes the theory of the determination of the heliocentric and geocentric coordinates of a body, moving in any orbit, when either the ecliptic or the equatorial system is used

XVI PROBLEMS

1 Interpret the angle N, equation (69), geometrically and show that n is simply a factor of proportionality

2 Suppose the ascending node be taken always as that one which is less than 180, and that the inclination varies from $-90°$ to $+90°$, discuss the changes which will be made in the equations (60) (76), and in particular, write the definitions of the Gaussian constants a, A, , C for this method of defining the elements

3 Interpret the Gaussian constants, defined by (74), geometrically

HISTORICAL SKETCH AND BIBLIOGRAPHY

The Problem of Two Bodies for spheres of finite size was first solved by Newton about 1686, and is given in the *Principia*, Book I, Section 11 The demonstration is geometrical The methods of the Calculus were cultivated with ardor in continental Europe at the beginning of the 18th century, but Newton's system of Mechanics did not find immediate acceptance, indeed, the French clung to the vortex theory of Descartes (1596—1650) until Voltaire, after his visit to London 1727, vigorously supported the Newtonian theory, 1728—1738 This, with the fact that the English continued to employ the geometrical methods of the *Principia*, delayed the analytical solution of the problem It was probably accomplished by Daniel Bernoulli in the memoir for which he received the prize from the French Academy in 1734, and it was certainly solved in detail by Euler in 1744 in his *Theoria motuum planetarum et cometarum* Since that time the modifications have been chiefly in the choice of variables in which the problem has been expressed

The solution of Kepler's equation naturally was first made by Kepler himself The next was by Newton in the *Principia* From a graphical construction involving the cycloid he was able to find very easily the approximate solution for the eccentric anomaly A very large number of analytical and graphical solutions have been discovered, nearly every prominent mathematician from Newton until the middle of the last century having given the subject more or less attention Among Americans, Professor Howe, of Denver, has given Kepler's equation the most study, and he has published several methods of solving it in the *Astronomische Nachrichten*. A bibliography containing references to 123 papers on Kepler's

equation is given in the *Bulletin Astronomique*, Jan 1900, and even this extended list is incomplete

The transformations of coordinates involve merely the solutions of spherical triangles, the treatment of which in a perfectly general form the mathematical world owes to Gauss (1777—1855), and which was intro duced into American Trigonometries by the late Professor Chauvenet of St Louis

The Problem of Two Bodies is treated in every work on Analytical Mechanics The reader will do well to consult further Tisserand's *Méc Cél* vol I, chapters VI and VII

CHAPTER VI

THE GENERAL INTEGRALS OF THE PROBLEM OF n BODIES

106 The Differential Equations of Motion Suppose the bodies are homogeneous in spherical layers, then they will attract each other as though their masses were at their centers Let m_1, m_2, , m_n represent their masses Let the coordinates of m_i referred to a fixed system of axes be x_i, y_i, z_i ($i = 1$, , n) Let $r_{i\,j}$ represent the distance between the centers of m_i and m_j Let k^2 represent a constant depending upon the units employed Then the components of force on m_1 along the x-axis are

$$-\frac{k^2 m_1 m_2}{r_{1\,2}^2}\,\frac{(x_1 - x_2)}{r_{1\,2}}, \quad -\frac{k^2 m_1 m_3}{r_{1\,3}^2}\,\frac{(x_1 - x_3)}{r_{1,3}}, \quad , -\frac{k^2 m_1 m_n}{r_{1\,n}^2}\,\frac{(x_1 - x_n)}{r_{1\,n}},$$

and the total force is their sum Therefore

$$m_1 \frac{d^2 x_1}{dt^2} = -k^2 m_1 \sum_{j=2}^{n} m_j \frac{(x_1 - x_j)}{r_{1\,j}^3},$$

and similar equations in y and z

There are similar equations for each body, the whole system being

$$(1)\quad \begin{cases} m_i \dfrac{d^2 x_i}{dt^2} = -k^2 m_i \displaystyle\sum_{j=1}^{n} m_j \dfrac{(x_i - x_j)}{r_{i\,j}^3}, \\ m_i \dfrac{d^2 y_i}{dt^2} = -k^2 m_i \displaystyle\sum_{j=1}^{n} m_j \dfrac{(y_i - y_j)}{r_{i\,j}^3}, \\ m_i \dfrac{d^2 z_i}{dt^2} = -k^2 m_i \displaystyle\sum_{j=1}^{n} m_j \dfrac{(z_i - z_j)}{r_{i\,j}^3}, \quad (i = 1, \quad , n), \ (j \neq i) \end{cases}$$

Each equation involves all of the $3n$ variables and the system must, therefore, be solved simultaneously There are $3n$ equations each of the second order, so that the problem is of order $6n$

These equations may be put in a simple and elegant form by the introduction of the potential function, which in this problem will be denoted by U instead of V The constant k^2 will be included in the potential In Chapter IV the potential V was defined by the integral

$V = \int \frac{dm}{\rho}$ In this case the system is composed of discrete masses, and the potential is

$$(2) \qquad U = \tfrac{1}{2}k^2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{m_i m_j}{r_{ij}}, \quad (i \neq j)$$

The partial derivative of U with respect to x_i is

$$\frac{\partial U}{\partial x_i} = k^2 m_i \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \frac{m_j}{r_{ij}} = -k^2 m_i \sum_{j=1}^{n} m_j \frac{(x_i - x_j)}{r_{ij}^3}, \quad (i \neq j),$$

and similar equations in y and z Therefore equations (1) may be written

$$(3) \qquad \begin{cases} m_i \dfrac{d^2 x_i}{dt^2} = \dfrac{\partial U}{\partial x_i}, \\ m_i \dfrac{d^2 y_i}{dt^2} = \dfrac{\partial U}{\partial y_i}, \\ m_i \dfrac{d^2 z_i}{dt^2} = \dfrac{\partial U}{\partial z_i}, \quad (i = 1, \quad , n) \end{cases}$$

107 The Six Integrals of the Motion of the Center of Mass The function U is independent of the choice of the coordinate axes since it depends upon the mutual distances of the bodies alone Therefore, if the origin is displaced parallel to the x-axis in the negative direction through a distance α the x-coordinate of every body will be increased by the quantity α, but the potential function will not be changed Let the fact that U is a function of all the x-coordinates be indicated by writing

$$U = U(x_1, x_2, \quad , x_n)$$

After the origin is displaced the x-coordinates become

$$x_i' = x_i + \alpha, \quad (i = 1, \quad , n)$$

The partial derivative of U with respect to α is

$$\frac{\partial U}{\partial \alpha} = \frac{\partial U}{\partial x_1'} \frac{\partial x_1'}{\partial \alpha} + \frac{\partial U}{\partial x_2'} \frac{\partial x'}{\partial \alpha} + \quad + \frac{\partial U}{\partial x_n'} \frac{\partial x_n'}{\partial \alpha}$$

But $\frac{\partial x_i'}{\partial \alpha} = 1$, $(i = 1, \quad , n)$, and $\frac{\partial U}{\partial \alpha} = 0$, as U does not involve α explicitly Therefore, dropping the accents and writing the corresponding equations in y and z,

$$\begin{cases} \sum_{i=1}^{n} \dfrac{\partial U}{\partial x_i} = 0, \\ \sum_{i=1}^{n} \dfrac{\partial U}{\partial y_i} = 0, \\ \sum_{i=1}^{n} \dfrac{\partial U}{\partial z_i} = 0 \end{cases}$$

Then equations (3) give

$$\begin{cases} \sum_{i=1}^{n} m_i \dfrac{d^2x_i}{dt^2} = 0, \\ \sum_{i=1}^{n} m_i \dfrac{d^2y_i}{dt^2} = 0, \\ \sum_{i=1}^{n} m_i \dfrac{d^2z_i}{dt^2} = 0 \end{cases}$$

These equations are at once integrable, and give

$$(4) \qquad \begin{cases} \sum_{i=1}^{n} m_i \dfrac{dx_i}{dt} = \alpha_1, \\ \sum_{i=1}^{n} m_i \dfrac{dy_i}{dt} = \beta_1, \\ \sum_{i=1}^{n} m_i \dfrac{dz_i}{dt} = \gamma_1, \end{cases}$$

where α_1, β_1, γ_1 are the constants of integration Integrating again, it follows that

$$(5) \qquad \begin{cases} \sum_{i=1}^{n} m_i x_i = \alpha_1 t + \alpha_2, \\ \sum_{i=1}^{n} m_i y_i = \beta_1 t + \beta_2, \\ \sum_{i=1}^{n} m_i z_i = \gamma_1 t + \gamma_2 \end{cases}$$

Let $\sum_{i=1}^{n} m_i = M$, and $\bar{x}$, $\bar{y}$, and $\bar{z}$ represent the coordinates of the center of mass, then, by Art 19,

$$(6) \qquad \begin{cases} \sum_{i=1}^{n} m_i x_i = M\bar{x}, \\ \sum_{i=1}^{n} m_i y_i = M\bar{y}, \\ \sum_{i=1}^{n} m_i z_i = M\bar{z} \end{cases}$$

Therefore, equations (5) may be written

$$(7) \qquad \begin{cases} M\bar{x} = \alpha_1 t + \alpha_2, \\ M\bar{y} = \beta_1 t + \beta_2, \\ M\bar{z} = \gamma_1 t + \gamma_2, \end{cases}$$

that is, the coordinates of the center of mass vary directly as the time From this it may be inferred that the center of mass moves

with uniform speed in a straight line Or otherwise, the velocity of the center of mass is

$$(8)\qquad \overline{V}=\sqrt{\left(\frac{d\bar{x}}{dt}\right)^2+\left(\frac{d\bar{y}}{dt}\right)^2+\left(\frac{d\bar{z}}{dt}\right)^2}=\frac{1}{M}\sqrt{\alpha_1+\beta_1^2+\gamma_1^2}=\text{constant}$$

Eliminating t from equations (7), it follows that

$$(9)\qquad \frac{M\bar{x}-\alpha_2}{\alpha_1}=\frac{M\bar{y}-\beta_2}{\beta_1}=\frac{M\bar{z}-\gamma_2}{\gamma_1},$$

which are the symmetrical equations of a straight line in three dimensions Equations (8) and (9) give the theorem

If n bodies are subject to no forces except their mutual attractions their center of mass moves in a straight line with uniform speed The special case $\overline{V}=0$ will arise if $\alpha_1=\beta_1=\gamma_1=0$

Since it is impossible to know any fixed point in space it is impossible to determine practically the six constants

The origin might now be transferred to the center of mass of the system, as it was in the Problem of Two Bodies, or, to the center of one of the bodies as it will be in Art 111, and the order of the problem reduced six units

108 The Three Integrals of Areas The potential function is not changed by a rotation of the axes Rotate the system of coordinates around the z-axis through the angle $-\phi$ and call the new coordinates x, y', and z' They are related to the old by the equations

$$(10)\qquad \begin{cases} x_i'=x_i\cos\phi-y_i\sin\phi,\\ y_i'=x_i\sin\phi+y_i\cos\phi,\\ z_i'=z_i,\quad (i=1,\ \cdots,n) \end{cases}$$

Since the function U is not changed by the rotation it does not contain ϕ explicitly, therefore, it follows that

$$(11)\qquad \frac{\partial U}{\partial\phi}=\sum_{i=1}^{n}\frac{\partial U}{\partial x_i'}\frac{\partial x_i'}{\partial\phi}+\sum_{i=1}^{n}\frac{\partial U}{\partial y_i'}\frac{\partial y_i'}{\partial\phi}+\sum_{i=1}^{n}\frac{\partial U}{\partial z_i'}\frac{\partial z_i'}{\partial\phi}=0$$

But from (10)

$$\frac{\partial x_i'}{\partial\phi}=-y_i',\quad \frac{\partial y_i'}{\partial\phi}=x_i',\quad \frac{\partial z_i'}{\partial\phi}=0,\quad (i=1,\ \cdots,n),$$

therefore (11) becomes

$$\sum_{i=1}^{n}\left(x_i'\frac{\partial U}{\partial y_i'}-y_i'\frac{\partial U}{\partial x_i'}\right)=0$$

Dropping the accents, which are of no further use, it follows as a consequence of (3) that

$$\begin{cases} \sum\limits_{i=1}^{n} m_i\left(x_i \dfrac{d^2y_i}{dt^2} - y_i \dfrac{d^2x_i}{dt^2}\right) = 0, \text{ and similarly,} \\ \sum\limits_{i=1}^{n} m_i\left(y_i \dfrac{d^2z_i}{dt^2} - z_i \dfrac{d^2y_i}{dt^2}\right) = 0, \\ \sum\limits_{i=1}^{n} m_i\left(z_i \dfrac{d^2x_i}{dt^2} - x_i \dfrac{d^2z_i}{dt^2}\right) = 0 \end{cases}$$

Each term in the summations may be integrated separately, giving

$$(12)\qquad \begin{cases} \sum\limits_{i=1}^{n} m_i\left(x_i \dfrac{dy_i}{dt} - y_i \dfrac{dx_i}{dt}\right) = c_1, \\ \sum\limits_{i=1}^{n} m_i\left(y_i \dfrac{dz_i}{dt} - z_i \dfrac{dy_i}{dt}\right) = c_2, \\ \sum\limits_{i=1}^{n} m_i\left(z_i \dfrac{dx_i}{dt} - x_i \dfrac{dz_i}{dt}\right) = c_3 \end{cases}$$

The parentheses are the projections of the areal velocities of the various bodies upon the three fundamental planes (Art 16) As it is impossible to determine any fixed point in space, so also it is impossible to determine any fixed direction in space, consequently it is impossible to determine practically the constants c_1, c_2, c_3 Yet, in this case it is customary to assume that the fixed stars, on the average, do not revolve in space, so that, by observing them, these constants can be determined It is evident, however, that there is no more reason for assuming that the stars do not revolve than there is for assuming that they are not drifting through space, each being a pure assumption without any possibility of proof or disproof But it is to be noted that, if these assumptions are granted, the constants c_1, c_2, and c_3 can be determined easily with a high degree of precision, while in the present state of observational Astronomy the constants of equations (4) cannot be found with any considerable accuracy

Let A_i, B_i, and C_i represent the projections of the areal velocity of m_i upon the xy, yz, and zx-planes respectively, then (12) may be written

$$\begin{cases} \sum\limits_{i=1}^{n} m_i \dfrac{dA_i}{dt} = c_1, \\ \sum\limits_{i=1}^{n} m_i \dfrac{dB_i}{dt} = c_2, \\ \sum\limits_{i=1}^{n} m_i \dfrac{dC_i}{dt} = c_3, \end{cases}$$

the integrals of which are

$$(13)\qquad \begin{cases} \sum_{i=1}^{n} m_i A_i = c_1 t + c_1', \\ \sum_{i=1}^{n} m_i B_i = c_2 t + c_2', \\ \sum_{i=1}^{n} m_i C_i = c_3 t + c_3' \end{cases}$$

Hence the theorem

The sums of the products of the masses and the projections of the areas described by the corresponding radii are proportional to the time, or, from (12), the sums of the products of the masses and the rates of the projections of the areas are constants

It is possible, as was first shown by Laplace, to direct the axes so that two of the constants in equations (12) will be zero, while the third becomes $\sqrt{c_1^2 + c_2^2 + c_3^2}$ This is the plane of maximum sum of the products of the masses and the rates of the projections of areas Its relations to the original fixed axes are defined by the constants c_1, c_2, c_3, and its position is, therefore, always the same On this account it was called the *invariable plane* by Laplace At present the invariable plane of the solar system is inclined to the ecliptic by about 2°, and the longitude of its ascending node is about 286 These figures are subject to some uncertainty because of our imperfect knowledge regarding the masses of some of the planets If the position of the plane were known with exactness it would possess some practical advantages over the ecliptic, which undergoes considerable variations, as a fundamental plane of reference It has been of great value in certain theoretical investigations*

109 The Energy Integral† Multiplying (3) by $\frac{dx_i}{dt}, \frac{dy_i}{dt}, \frac{dz_i}{dt}$ respectively, adding and summing with respect to i, it follows that

$$(14)\qquad \sum_{i=1}^{n} m_i \left\{\frac{d^2x_i}{dt^2}\frac{dx_i}{dt} + \frac{d^2y_i}{dt^2}\frac{dy_i}{dt} + \frac{d^2z_i}{dt^2}\frac{dz_i}{dt}\right\} = \sum_{i=1}^{n} \left\{\frac{\partial U}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial U}{\partial y_i}\frac{dy_i}{dt} + \frac{\partial U}{\partial z_i}\frac{dz_i}{dt}\right\}$$

U is a function of the $3n$ variables x_i, y_i, z_i, alone, therefore the right

* See memoirs by Jacobi, *Journal de Math* vol IX, Tisserand, *Méc Cél* vol I chap XXV, Poincaré, *Les Methodes Nouvelles de la Mec Cél* vol I p 39

† This is very frequently called the *Vis Viva* integral

member of (14) is the total derivative of U with respect to t Integrating both members of the equation, it is found that

$$(15) \qquad \tfrac{1}{2}\sum_{i=1}^{n} m_i \left\{\left(\frac{dx_i}{dt}\right)^2 + \left(\frac{dy_i}{dt}\right)^2 + \left(\frac{dz_i}{dt}\right)^2\right\} = U + h$$

The left member of this equation is the kinetic energy of the whole system, and the right member is the potential function plus a constant

Let the potential energy of one configuration of a system with respect to another configuration be defined as the amount of work required to change it from one to the other If two bodies attract each other according to the law of the inverse squares the force existing between them is $\frac{k^2 m_i m_j}{r^2_{i,j}}$ The amount of work done in changing their distance apart from $r^{(0)}_{i,j}$ to $r_{i,j}$ is

$$(16) \qquad W_{i,j} = k^2 m_i m_j \int_{r^{(0)}_{i,j}}^{r_{i,j}} \frac{dr_{i,j}}{r^2_{i,j}} = k^2 m_i m_j \left(\frac{1}{r^{(0)}_{i,j}} - \frac{1}{r_{i,j}}\right)$$

If the bodies are at an infinite distance from each other at the start, then $r^{(0)}_{i,j} = \infty$, and (16) becomes

$$-W_{i,j} = \frac{k^2 m_i m_j}{r_{i,j}},$$

hence

$$U = -\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} W_{i,j}$$

Therefore, U is the negative of the potential energy of the whole system with respect to the infinite separation of the bodies as the original configuration Hence (15) gives the theorem

In a system of n bodies subject to no forces except their mutual attractions the sum of the kinetic and potential energies is a constant

The only case in which this result would be open to question would be that in which some of the velocities become infinite But this could happen only if some $r_{i,j}$ should become equal to zero If this should happen the differential equations would cease to have a meaning at that instant, and a special discussion would be required to show that the integral held true beyond that time

110 The Question of New Integrals Ten of the whole $6n$ integrals which are required in order to solve the problem completely have been found These ten integrals are the only ones known, and the question arises whether any more of certain types exist

In a profound memoir in the *Acta Mathematica*, vol XI, Professor Bruns has demonstrated that there are no new algebraic integrals Poincaré has demonstrated in his prize memoir in the *Acta Mathematica*, vol XIII, and again with some additions in *Les Méthodes Nouvelles de la Mécanique Céleste*, chap V, that the Problem of Three Bodies admits no new uniform transcendental integrals, even when the masses of two of the bodies are very small compared to that of the third

XVII PROBLEMS

1 Write equations (1) when the force varies inversely as the nth power of the distance For what values of n do the equations all become independent? The Problem of n Bodies can be completely solved for this law of force, show that the orbits with respect to the center of mass of the system are all ellipses with this point as center Show that the orbit of any body with respect to any other is also a central ellipse, and that the same is true for the motion of any body with respect to the center of mass of any sub-group of the whole system Show that the periods are all equal

2 What will be the definition of the potential function when the force varies inversely as the nth power of the distance?

3 Derive the equations immediately preceding (4) directly from equations (1)

4 Prove that the theorem regarding the motion of the center of mass holds when the force varies as any power of the distance

5 Derive the equations immediately preceding (12) directly from equations (1), and show that they hold when the force varies as any power of the distance

6 Any plane through the origin may be changed into any other plane through the origin by a rotation around each of two of the coördinate axes Transform equations (12) by successive rotations around two of the axes, and show that the angles of rotation may be chosen so that two of the constants, to which the functions of the new coordinates similar to (12) are equal, are zero, and that the third is $\sqrt{c_1^2+c_2^2+c_3^2}$ (This is the method used by Laplace to prove the existence of the invariable plane)

7 Why are equations (13) not to be regarded as integrals of the differential equations (1), thus making the whole number of integrals thirteen?

111 Transfer of the Origin to the Sun Nothing is known of the absolute motions of the planets, since the observations furnish information regarding only their relative positions, or their positions with respect to the sun It is true that it is known that the solar system is moving toward the constellation Hercules, but it must be remembered that this motion is only with respect to the fixed stars The problem for the student of Celestial Mechanics is to determine the relative positions of the members of the solar system, or, in particular, to determine the positions of the planets with respect to the sun To do this it is advantageous to transfer the origin to the sun, and to employ the resulting differential equations

Suppose m_n is the sun and take its center as the origin, and let the coordinates of the body m_i referred to the new system be x_i', y_i', z_i' then the old coordinates are expressed in terms of the new by the equations

$$x_i = x_i' + x_n, \quad y_i = y_i' + y_n, \quad z_i = z_i' + z_n, \qquad (i = 1, \quad , n-1)$$

Since the differences of the old variables are equal to the corresponding differences of the new, it follows that

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_i'}, \quad \frac{\partial U}{\partial y_i} = \frac{\partial U}{\partial y_i'}, \quad \frac{\partial U}{\partial z_i} = \frac{\partial U}{\partial z_i'}, \qquad (i = 1, \quad , n-1)$$

As a consequence of these transformations equations (3) become

$$(17) \qquad \begin{cases} \dfrac{d^2x_i'}{dt^2} + \dfrac{d^2x_n}{dt^2} = \dfrac{1}{m_i}\dfrac{\partial U}{\partial x_i'}, \\ \dfrac{d^2y_i'}{dt^2} + \dfrac{d^2y_n}{dt^2} = \dfrac{1}{m_i}\dfrac{\partial U}{\partial y_i'}, \\ \dfrac{d^2z_i'}{dt^2} + \dfrac{d^2z_n}{dt^2} = \dfrac{1}{m_i}\dfrac{\partial U}{\partial z_i'}, \quad (i = 1, \quad , n-1) \end{cases}$$

Since the origin is at m_n, $x_n' = y_n' = z_n' = 0$, and the first equation of (1) gives, putting $i = n$,

$$(18) \qquad \frac{d^2x_n}{dt^2} = \frac{k^2m_1x_1'}{r^3_{1\,n}} + \frac{k^2m_2x_2'}{r^3_{2\,n}} + \quad + \frac{k^2m_{n-1}x'_{n-1}}{r^3_{n-1\,n}} = k^2\sum_{j=1}^{n-1}\frac{m_jx_j'}{r^3_{j\,n}}$$

This equation, with the corresponding ones in y and z, substituted in (17) completes the transformation to the new variables, but it will be advantageous to combine the terms in another manner so that the terms coming from the attraction of the sun shall be separate from the others The differential equations will be written for the body m_1, from which the others may be formed by permuting the subscripts

Let

$$U = k^2 m_n \sum_{i=1}^{n-1} \frac{m_i}{r_{i\,n}} + \tfrac{1}{2} k \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} \frac{m_i m_j}{r_{i\,j}}, \qquad (i \neq j),$$

or,

$$(19) \qquad U = k^2 m_n \sum_{j=1}^{n-1} \frac{m_j}{r_{j\,n}} + U'$$

Making the substitutions (18) and (19) in equations (17), they become

$$(20) \quad \begin{cases} \dfrac{d\ x_1'}{dt^2} + k\ (m_1 + m_n) \dfrac{x_1'}{r^3_{1\,n}} = \dfrac{1}{m_1} \dfrac{\partial U'}{\partial x_1'} - k^2 \sum\limits_{j=2}^{n-1} \dfrac{m_j x_j'}{r^3_{j\,n}}, \\ \dfrac{d^2 y_1'}{dt^2} + k^2 (m_1 + m_n) \dfrac{y_1'}{r^3_{1\,n}} = \dfrac{1}{m_1} \dfrac{\partial U'}{\partial y_1'} - k^2 \sum\limits_{j=2}^{n-1} \dfrac{m_j y_j'}{r^3_{j\,n}}, \\ \dfrac{d^2 z_1'}{dt^2} + k\ (m_1 + m_n) \dfrac{z_1'}{r^3_{1\,n}} = \dfrac{1}{m_1} \dfrac{\partial U'}{\partial z_1'} - k^2 \sum\limits_{j=2}^{n-1} \dfrac{m_j z_j'}{r^3_{j\,n}} \end{cases}$$

Let

$$R_{1\,j} = k \left\{ \frac{1}{r_{1\,j}} - \frac{x_1' x_j' + y_1' y_j' + z_1' z_j'}{r^3_{j\,n}} \right\},$$

then, equations (20) may be written

$$(21) \quad \begin{cases} \dfrac{d\ x_1'}{dt} + k^2 (m_1 + m_n) \dfrac{x_1'}{r^3_{1\,n}} = \sum\limits_{j=2}^{n-1} m_j \dfrac{\partial R_{1\,j}}{\partial x_1'}, \\ \dfrac{d^2 y_1'}{dt} + k\ (m_1 + m_n) \dfrac{y_1'}{r^3_{1\,n}} = \sum\limits_{j=2}^{n-1} m_j \dfrac{\partial R_{1\,j}}{\partial y_1'}, \\ \dfrac{d^2 z_1'}{dt} + k^2 (m_1 + m_n) \dfrac{z_1'}{r^3_{1\,n}} = \sum\limits_{j=2}^{n-1} m_j \dfrac{\partial R_{1\,j}}{\partial z_1'} \end{cases}$$

Let the accents, which have become useless, be dropped, and, in order to derive the general equations corresponding to (21), let

$$(22) \qquad R_{i\,j} = k \left\{ \frac{1}{r_{i\,j}} - \frac{x_i x_j + y_i y_j + z_i z_j}{r^3_{j\,n}} \right\}, \qquad (i \neq j)$$

Then, the general equations for relative motion are

$$(23) \quad \begin{cases} \dfrac{d^2 x_i}{dt} + k\ (m_i + m_n) \dfrac{x_i}{r^3_{i\,n}} = \sum\limits_{j=1}^{n-1} m_j \dfrac{\partial R_{i\,j}}{\partial x_i}, & (i \neq j), \\ \dfrac{d\ y_i}{dt^2} + k^2 (m_i + m_n) \dfrac{y_i}{r^3_{i\,n}} = \sum\limits_{j=1}^{n-1} m_j \dfrac{\partial R_{i\,j}}{\partial y_i}, & \\ \dfrac{d^2 z_i}{dt} + k^2 (m_i + m_n) \dfrac{z_i}{r_{i\,n}} = \sum\limits_{j=1}^{n-1} m_j \dfrac{\partial R_{i\,j}}{\partial z_i}, & (i = 1, \quad , n-1) \end{cases}$$

112 Dynamical Meaning of the Equations In order to perceive easily the meaning of the equations, suppose that there are but three bodies, m_1, m_2, and m_n Suppose m_n is the sun, let its mass

equal unity, and let the distances from it to m_1 and m_2 be r_1 and r_2 respectively. Then equations (23) are, in full,

$$
(24) \quad \left\{
\begin{aligned}
\frac{d^2x_1}{dt^2} + k^2(1+m_1)\frac{x_1}{r_1^3} &= k^2m_2\frac{\partial}{\partial x_1}\left\{\frac{1}{r_{1,2}} - \frac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}\right\}, \\
\frac{d^2y_1}{dt^2} + k^2(1+m_1)\frac{y_1}{r_1^3} &= k^2m_2\frac{\partial}{\partial y_1}\left\{\frac{1}{r_{1,2}} - \frac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}\right\}, \\
\frac{d^2z_1}{dt^2} + k^2(1+m_1)\frac{z_1}{r_1^3} &= k^2m_2\frac{\partial}{\partial z_1}\left\{\frac{1}{r_{1,2}} - \frac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}\right\}, \\
\frac{d^2x_2}{dt^2} + k^2(1+m_2)\frac{x_2}{r_2^3} &= k^2m_1\frac{\partial}{\partial x_2}\left\{\frac{1}{r_{2,1}} - \frac{x_2x_1+y_2y_1+z_2z_1}{r_1^3}\right\}, \\
\frac{d^2y_2}{dt^2} + k^2(1+m_2)\frac{y_2}{r_2^3} &= k^2m_1\frac{\partial}{\partial y_2}\left\{\frac{1}{r_{2,1}} - \frac{x_2x_1+y_2y_1+z_2z_1}{r_1^3}\right\}, \\
\frac{d^2z_2}{dt^2} + k^2(1+m_2)\frac{z_2}{r_2^3} &= k^2m_1\frac{\partial}{\partial z_2}\left\{\frac{1}{r_{2,1}} - \frac{x_2x_1+y_2y_1+z_2z_1}{r_1^3}\right\}
\end{aligned}
\right.
$$

If m_2 were zero the first three equations would be independent of the second three, and (24) would then be the equations for the relative motions of the two bodies m_1 and m, and could be integrated. All the variations from the purely elliptical motion arise from the presence of the right members, which are, in the first three equations, the partial derivatives of $R_{1,2}$ with respect to the variables x_1, y_1, and z_1 respectively. On this account $m_2R_{1,2}$ is called the *perturbative function*.

The partial derivatives of the first terms of the right members are

$$
-k^2m_2\frac{(x_1-x_2)}{r^3_{1,2}}, \quad -k^2m_2\frac{(y_1-y_2)}{r^3_{1,2}}, \quad -k^2m_2\frac{(z_1-z_2)}{r^3_{1,2}},
$$

which are the components of acceleration of m_1 due to the attraction of m_2. The partial derivatives of the second terms are

$$
-k^2m_2\frac{x_2}{r_2^3}, \quad -k^2m_2\frac{y_2}{r_2^3}, \quad -k^2m_2\frac{z_2}{r_2^3},
$$

which are the negatives of the components of the acceleration of the sun due to the attraction of m_2. Therefore the right members of the first three equations of (24) are the differences of the components of acceleration of m_1 and the sun due to the attraction of m_2. Similarly, the right members of the last three equations are the differences of the components of the acceleration of m_2 and the sun due to the attraction of m_1. If two bodies are subject to equal parallel accelerations their relative positions will not be changed. The differences of their accelerations are due to the disturbing forces, and measure these disturbances. The right members of (24) are, therefore, exactly those parts of the accelerations due to the disturbing forces.

If there are $n-2$ disturbing bodies the right members are the sums of terms depending upon the bodies $m_3, \ldots, m_{n-1}$, similar to the right members of (24), which depend upon m alone; or, in other words, the whole resultants of the disturbing accelerations are equal to the sums of the parts arising from the action of the separate disturbing bodies.

113. The Order of the System of Equations. The order of the system of equations (23) is $6n-6$, instead of $6n$ as (1) was in the case of absolute motion. In the absolute motion ten integrals were found which reduced the problem to order $6n-10$. Six of these related to the motion of the center of mass, three, to the areal velocities, and one to the energy of the system. In the present case but four integrals, the three integrals of areas and the energy integral, can be found, which leaves the problem of order $6n-10$ also.

The problem can be reduced to the order $6n-6$ by using the integrals for the center of mass directly. In particular, consider the differential equations for the bodies $m_1, m_2, \ldots, m_{n-1}$. In the original equations they involve the coordinates of m_n, but these quantities may be eliminated by means of equations (5).

If the origin is taken at the center of mass

$$\sum_{i=1}^{n} m_i x_i = 0, \quad \sum_{i=1}^{n} m_i y_i = 0, \quad \sum_{i=1}^{n} m_i z_i = 0,$$

and the elimination becomes particularly simple. Or, because of these linear homogeneous relations, the n variables of each set may be expressed linearly and homogeneously in terms of $n-1$ new variables. Thus

$$\begin{aligned} x_1 &= a_{11}\xi_1 + a_{12}\xi_2 + \cdots + a_{1\,n-1}\xi_{n-1}, \\ x_2 &= a_{21}\xi_1 + a_{22}\xi_2 + \cdots + a_{2\,n-1}\xi_{n-1}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n &= a_{n1}\xi_1 + a_{n2}\xi_2 + \cdots + a_{n\,n-1}\xi_{n-1}, \end{aligned}$$

and similar sets of equations for y and z. The coefficients a_{ij} are arbitrary constants except that they must be chosen so that every determinant of the matrix of the substitutions shall be distinct from zero; for, otherwise, a linear relation would exist among the ξ_i. These conditions may be chosen so that the transformed equations preserve a symmetrical form. This method was employed by Jacobi in an important memoir entitled, *Sur l'élimination des nœuds dans le problème des trois corps* (*Journal de Math.* vol. IX., 1844), and by Radau in a memoir entitled, *Sur une transformation des équations différentielles de la Dynamique* (*Annales de l'École Normale*, 1st series, vol. V.)

XVIII PROBLEMS

1 Make the transformation $x_i = x_i' + x_n$ in the integrals (12) and (15), and eliminate x_n, y_n, z_n, $\frac{dx_n}{dt}$, $\frac{dy_n}{dt}$, and $\frac{dz_n}{dt}$ by means of equations (4) and (5) Prove that the resulting expressions are four integrals of equations (23)

2 Derive equations (23) directly by taking the origin at m_n, without first making use of the fixed axes

3 The equations (23) are not symmetrical, since each body requires a different perturbative function $R_{i,j}$ in the right members Construct the corresponding system of differential equations where the motion of m_{n-1} is referred to a rectangular system of axes with the origin at m_n, the motion of m_{n-2} to a parallel system of axes with origin at the center of mass of m_n and m_{n-1}, the motion of m_{n-3} to a parallel system of axes with the origin at the center of mass of m_n, m_{n-1}, and m_{n-2}, and continue in this way Show that the results are the symmetrical equations

$$
\begin{aligned}
&\frac{\mu_n}{\mu_{n-1}} m_{n-1} \frac{d^2 x_{n-1}}{dt^2} = \frac{\partial U}{\partial x_{n-1}}, \quad \mu_n = m_n, \quad \mu_{n-1} = m_{n-1} + m_n, \\
&\frac{\mu_{n-1}}{\mu_{n-2}} m_{n-2} \frac{d^2 x_{n-2}}{dt^2} = \frac{\partial U}{\partial x_{n-2}}, \quad \mu_{n-2} = m_{n-2} + m_{n-1} + m_n, \\
&\qquad \cdots \\
&\frac{\mu_2}{\mu_1} m_1 \frac{d^2 x_1}{dt^2} = \frac{\partial U}{\partial x_1}, \quad \mu_1 = m_1 + m_2 + \cdots + m_n,
\end{aligned}
$$

and similar equations in y and z, where

$$
\begin{aligned}
U = k^2 m_n & \left(\frac{m_{n-1}}{r_{n,\,n-1}} + \frac{m_{n-2}}{r_{n,\,n-2}} + \cdots + \frac{m_1}{r_{n\,1}}\right) \\
&+ k^2 m_{n-1} \left(\frac{m_{n-2}}{r_{n-1\;n-2}} + \frac{m_{n-3}}{r_{n-1\;n-3}} + \cdots + \frac{m_1}{r_{n-1\;1}}\right) \\
&\qquad \cdots \\
& k^2 m_2 \frac{m_1}{r_{1\,2}}
\end{aligned}
$$

(These equations are the same as found by Radau from a different standpoint in the memoir cited in Art 113 They have been employed by Tisserand in a very elegant demonstration of Poisson's theorem of the invariability of the major axes of the planets' orbits up to perturbations of the second order inclusive with respect to the masses Poincaré has generally used this system in his researches in the Problem of Three Bodies)

4 Derive the differential equations corresponding to (23) in polar coördinates

$$
\begin{cases}
\dfrac{d^2 r_{j\,n}}{dt^2} - r_{j\,n} \cos^2\phi_j \left(\dfrac{d\theta_j}{dt}\right)^2 - r_{j,\,n}\left(\dfrac{d\phi_j}{dt}\right)^2 = -\dfrac{k^2(m_j + m_n)}{r^2_{j\,n}} + \sum\limits_{i=1}^{n-1} m_i \dfrac{\partial R_{j\,i}}{\partial r_{i,\,n}}, \\[2ex]
\dfrac{d}{dt}\left(r^2_{j\,n} \cos^2\phi_j \dfrac{d\theta_j}{dt}\right) = \sum\limits_{i=1}^{n-1} m_i \dfrac{\partial R_{j\,i}}{\partial \theta_j}, \\[2ex]
\dfrac{d}{dt}\left(r^2_{j\,n} \dfrac{d\phi_j}{dt}\right) + r^2_{j\,n} \sin\phi_j \cos\phi_j \left(\dfrac{d\theta_j}{dt}\right)^2 = \sum\limits_{i=1}^{n-1} m_i \dfrac{\partial R_{j,\,i}}{\partial \phi_j}, \\[2ex]
(j = 1, \cdots, n-1), \quad (i \neq j)
\end{cases}
$$

HISTORICAL SKETCH AND BIBLIOGRAPHY

The investigations in the Problem of n Bodies are of two classes first, those which lead to general theorems holding in every system, and second, those which give good approximations for a certain length of time in particular systems, such as the solar system Investigations of the second class are known as theories of perturbations, the discussion of which will be given in another chapter

The first general theorems are regarding the motion of the center of mass, and were given by Newton in the *Principia* The ten integrals were known by Euler, and the theorems to which they lead The next general result was the proof of the existence of, and the discussion of the properties of, the invariable plane of Laplace in 1784 In the winter semester of 1842–43 Jacobi gave a course of lectures in the University of Königsberg on Dynamics In this course he gave the results of some very important investigations on the integration of the differential equations which arise in Mechanics In all cases where the forces depend upon the coordinates alone, and where a potential function exists, conditions which are fulfilled in the Problem of n Bodies, he proved that if all the integrals except two have been found the last two can always be found He also showed, in extending some investigations of Sir William Rowan Hamilton, that the problem is reduced to that of solving a partial differential equation whose order is one half as great as that of the original system Jacobi's lectures are published in the supplementary volume to his collected works They are of great importance in themselves, as well as being an absolutely necessary prerequisite to the reading of the epoch making memoirs of Poincaré, and they should be accessible to every student of Celestial Mechanics

It is a question of the highest interest whether the motions of the members of such a system as the sun and planets are purely periodic Professor Newcomb has shown in an important memoir published in the *Smithsonian Contributions to Knowledge*, December 1874, that the differential equations can be formally satisfied by purely periodic series He did not, however, prove the convergence of these series, and, indeed, Poincaré has shown in *Les Méthodes Nouvelles*, chaps IX and XII, that they are in general divergent

As was stated in Art 110, Bruns has proved in the *Acta Mathematica* vol XI that there are no new algebraic integrals, and Poincaré, in the *Acta Mathematica*, vol XIII, that there are no new uniform transcendental integrals even when the masses of all the bodies except one are very small

For further reading regarding the general differential equations in different sets of variables the student will do well to consult Tisserand's *Mécanique Céleste*, vol I chapters III IV and V

CHAPTER VII

THE PROBLEM OF THREE BODIES

114. Problem Considered. Certain theorems in the Problem of Three Bodies have been established with mathematical rigor when the mutual coordinates and the components of velocity fulfill certain conditions. While these cases have not been found in nature, there are nevertheless some applications of the results obtained, and the processes employed are mathematically elegant and lead to most interesting conclusions. This chapter will contain such of these results as fall within the scope of this work, reserving the theories of perturbations, by means of which the positions of the heavenly bodies are predicted, to subsequent chapters.

The first part will be devoted to a discussion of some of the properties of motion of an infinitesimal body when it is attracted by two finite bodies revolving in circles around their center of mass, and will include the proof of the existence of certain particular solutions in which the distances of the infinitesimal body from the finite bodies are constants. The second part will be devoted to an exposition of the method of finding particular solutions of the motion of three finite bodies such that the ratios of their mutual distances are constants. These solutions include the former, but the discoverable properties of motion are so much fewer, and are obtained with so much more difficulty, that it is advisable to divide the discussion into two parts.

These particular solutions of the Problem of Three Bodies were given for the first time by Lagrange in a prize memoir in 1772. The method adopted here is radically different from that employed by him, and lends itself much more readily to a generalization to the case where a larger number of bodies is involved. But, on the other hand, the reduction of the order of the problem by one unit, which was a very interesting feature of Lagrange's memoir, is not accomplished by this

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method However, as it has not been possible to make any use of this reduction, it has not been of any practical importance

Mathematically speaking, an *infinitesimal body* is such an one that it is attracted by finite masses according to the Newtonian law of gravitation, but does not attract them Physically speaking, it is a body of such a small mass that it will disturb the motion of finite bodies less than an arbitrary assigned amount, however small, during any arbitrary assigned time, however long To actually determine a small mass fulfilling these conditions it is only necessary to make it so small that its whole attraction, which is always greater than its disturbing force, on one of the large bodies, if placed at the minimum distance possible, would move the large body less than the assigned small distance in the assigned time

Motion of Infinitesimal Body

115 The Differential Equations of Motion Suppose the system consists of two finite bodies revolving in circles around their common center of mass, and of an infinitesimal body subject to their attraction Let the unit of mass be chosen so that the sum of the masses of the finite bodies shall be unity, then they may be represented by $1-\mu$ and μ, where the notation is chosen so that $\mu \leqq \frac{1}{2}$ Let the unit of distance be chosen so that the constant distance between the finite bodies shall be unity Let the unit of time be chosen so that k^2 shall equal unity Let the origin of coordinates be taken at the center of mass of the finite bodies, and let the direction of the axes be chosen so that the $\xi\eta$-plane is the plane of their motion Let the coordinates of $1-\mu$, μ, and the infinitesimal body be ξ_1, η_1, 0, ξ_2, η_2, 0, and ξ, η, ζ respectively, and

$$r_1 = \sqrt{(\xi-\xi_1)^2 + (\eta-\eta_1)^2 + \zeta^2}, \quad r_2 = \sqrt{(\xi-\xi_2)^2 + (\eta-\eta_2)^2 + \zeta^2}$$

Then the differential equations of motion for the infinitesimal body are

$$(1) \qquad \begin{cases} \dfrac{d^2\xi}{dt^2} = -(1-\mu)\dfrac{(\xi-\xi_1)}{r_1^3} - \mu\dfrac{(\xi-\xi_2)}{r_2^3}, \\[2ex] \dfrac{d^2\eta}{dt^2} = -(1-\mu)\dfrac{(\eta-\eta_1)}{r_1^3} - \mu\dfrac{(\eta-\eta_2)}{r_2^3}, \\[2ex] \dfrac{d^2\zeta}{dt^2} = -(1-\mu)\dfrac{\zeta}{r_1^3} - \mu\dfrac{\zeta}{r_2^3} \end{cases}$$

As a consequence of the way the units have been chosen the mean angular motion of the finite bodies is

$$n = k\,\frac{\sqrt{(1-\mu)+\mu}}{a^{\frac{3}{2}}} = 1$$

Refer the motion of the bodies to a new system of axes having the same origin as the old, and rotating in the $\xi\eta$-plane in the direction in which the finite bodies move with the uniform angular velocity unity The coordinates in the new system are defined by the equations

$$
(2) \qquad \begin{cases} \xi = x \cos t - y \sin t, \\ \eta = x \sin t + y \cos t, \\ \zeta = z, \end{cases}
$$

and similar equations for the letters with subscripts 1 and 2 Computing the second derivatives of (2) and substituting in (1), it is found that

$$
(3) \begin{cases} \left\{\frac{d x}{dt^2} - 2\frac{dy}{dt} - x\right\} \cos t - \left\{\frac{d y}{dt^2} + 2\frac{dx}{dt} - y\right\} \sin t \\ = -\left\{(1-\mu)\frac{(x-x_1)}{r_1^3} + \mu\frac{(a-x_2)}{r^3}\right\} \cos t + \left\{(1-\mu)\frac{(y-y_1)}{r_1^3} + \mu\frac{(y-y_2)}{r^3}\right\} \sin t, \\ \left\{\frac{d^2x}{dt} - 2\frac{dy}{dt} - x\right\} \sin t + \left\{\frac{d y}{dt} + 2\frac{dx}{dt} - y\right\} \cos t \\ = -\left\{(1-\mu)\frac{(x-x_1)}{r_1^3} + \mu\frac{(x-x_2)}{r_2^3}\right\} \sin t - \left\{(1-\mu)\frac{(y-y_1)}{r_1^3} + \mu\frac{(y-y_2)}{r_2^3}\right\} \cos t, \\ \frac{d^2z}{dt^2} = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \end{cases}
$$

Multiply the first two equations by $\cos t$ and $\sin t$ respectively, then by $-\sin t$ and $\cos t$, and add, the results are

$$
\begin{cases} \frac{d^2x}{dt^2} - 2\frac{dy}{dt} = x - (1-\mu)\frac{(x-x_1)}{r_1^3} - \mu\frac{(x-x_2)}{r_2^3}, \\ \frac{d y}{dt^2} + 2\frac{dx}{dt} = y - (1-\mu)\frac{(y-y_1)}{r_1^3} - \mu\frac{(y-y\ )}{r_2^3}, \\ \frac{d z}{dt^2} \qquad = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \end{cases}
$$

The position of the axes may be so taken at the origin of time that the x-axis will continually pass through the centers of the finite bodies, then $y_1 = 0$, $y_2 = 0$, and the equations become

$$
(4) \qquad \begin{cases} \frac{d^2x}{dt} - 2\frac{dy}{dt} = x - (1-\mu)\frac{(x-x_1)}{r_1^3} - \mu\frac{(x-x_2)}{r_2^3}, \\ \frac{d^2y}{dt^2} + 2\frac{dx}{dt} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}, \\ \frac{d^2z}{dt^2} \qquad = -(1-\mu)\frac{z}{r_1^3} - \mu\frac{z}{r_2^3} \end{cases}
$$

These are the differential equations of motion of the infinitesimal body referred to axes rotating so that the finite bodies always lie on the x-axis. They have the important property of involving the coordinates of the infinitesimal body alone as variables, the coordinates of the finite bodies having become constants as a consequence of the particular manner in which the axes are rotated.

The general problem of determining the motion of the infinitesimal body is of the sixth order; if it moves in the plane of motion of the finite bodies, the problem is of the fourth order.

116. Jacobi's Integral. Equations (4) admit an integral which was first given by Jacobi in *Comptes Rendus de l'Académie des Sciences de Paris*, vol. III. p. 59, and which has been discussed by Hill in the first of his celebrated papers on the Lunar Theory, *The American Journal of Mathematics*, vol. I. p. 18, and again by Darwin in his memoir on Periodic Orbits in *Acta Mathematica*, vol. XXI. p. 102. Let

$$(5) \qquad U = \tfrac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2},$$

then equations (4) may be written

$$(6) \qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = \dfrac{\partial U}{\partial x}, \\[2ex] \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = \dfrac{\partial U}{\partial y}, \\[2ex] \dfrac{d^2z}{dt^2} \qquad\quad = \dfrac{\partial U}{\partial z}. \end{cases}$$

If these equations are multiplied by $2\dfrac{dx}{dt}$, $2\dfrac{dy}{dt}$, and $2\dfrac{dz}{dt}$ respectively, and added, they may be integrated, since U is a function of x, y, and z alone, and give

$$(7) \quad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = V^2 = 2U - C$$

$$\equiv x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C.$$

Five integrals more are required in order to completely solve the problem. If the infinitesimal body moved in the xy-plane only three would remain to be found, the last two of which could be obtained by Jacobi's *last multiplier* *, if the first one were found. Thus it appears

* Developed in *Vorlesungen über Dynamik*, supplementary volume to Jacobi's collected works.

that only one new integral is needed for the complete solution of this special problem in the plane* But Bruns has proved in *Acta Mathematica*, vol XI, that no new algebraic integrals exist, and Poincaré has proved in *Les Méthodes Nouvelles de la Mécanique Céleste*, vol I, chap V, that there are no new uniform transcendental integrals, even when the mass of one of the finite bodies is very small compared to that of the other These demonstrations are entirely outside the scope of this work and cannot be reproduced here

117 The Surfaces of Zero Relative Velocity† Equation (7) is a relation between the square of the velocity and the coordinates of the infinitesimal body referred to the rotating axes Therefore, when the constant of integration C has been determined numerically by the initial conditions, equation (7) determines the velocity with which the infinitesimal body will move, if at all, at all points of the rotating space and conversely, for a given velocity, equation (7) gives the locus of those points of relative space where alone the infinitesimal body can be In particular, if V is put equal to zero in this equation it will define the surfaces at which the velocity will be zero On one side of these surfaces the velocity will be real and on the other side imaginary, or, in other words, it is possible for the body to move on one side, and impossible for it to move on the other While it will not be possible to say in any except very particular cases what the orbit will be, yet the partition of relative space will show in what portions the infinitesimal body may move and in what portions it cannot

In order to prove that if V^2 is positive on one side of a surface of zero relative velocity it is negative on the other, suppose x_0, y_0, z_0 are the coordinates of any point on the surface, and that $x_0 + \Delta x$, $y_0 + \Delta y$, $z_0 + \Delta z$ are the coordinates of a point P near it If the point P is on the normal to the surface at the point (x_0, y_0, z_0), the increments of the coordinates, Δx, Δy, Δz, will be proportional to the direction cosines of the normal Therefore if

$$V^2 = F(x, y, z) = 0$$

is the equation of the surface, and if P is on the normal, then

$$\Delta x = \rho \frac{\partial F}{\partial x_0}, \quad \Delta y = \rho \frac{\partial F}{\partial y_0}, \quad \Delta z = \rho \frac{\partial F}{\partial z_0},$$

* Hill put his special equations in such a form that they would be reduced to quadratures if a single variable were expressed in terms of the time *American Journal of Mathematics*, vol I p 16

† First discussed by Hill in his *Lunar Theory*, *The American Journal of Mathematics*, vol I, and again, for motion in the xy plane, by Darwin in his *Periodic Orbits*, in *Acta Mathematica*, vol XXI

where ρ is the distance from the surface to P, being positive on one side of the surface and negative on the other. The expression for V^2 becomes

$$V^2 = F\left(x_0 + \rho\frac{\partial F}{\partial x_0},\ y_0 + \rho\frac{\partial F}{\partial y_0},\ z_0 + \rho\frac{\partial F}{\partial z_0}\right),$$

which, developed by Taylor's formula, gives

$$V^2 = F(x_0, y_0, z_0) + \rho\left[\left(\frac{\partial F}{\partial x_0}\right)^2 + \left(\frac{\partial F}{\partial y_0}\right)^2 + \left(\frac{\partial F}{\partial z_0}\right)^2\right] + \text{higher terms in } \rho.$$

But

$$F(x_0, y_0, z_0) = 0,$$

because x_0, y_0, z_0 is in the surface of zero velocity.

Suppose x_0, y_0, z_0 is an ordinary point of the surface; then $\frac{\partial F}{\partial x_0}$, $\frac{\partial F}{\partial y_0}$, $\frac{\partial F}{\partial z_0}$ have determinate values not all zero, and, for sufficiently small values of ρ, V^2 has the same sign as ρ, because ρ may be taken so small that the sign of the function is the same as that of the first term of its expansion. Therefore, *if V^2 is positive on one side of a surface of zero relative velocity in the vicinity of an ordinary point, it is negative on the opposite side.*

The equation of the surfaces of zero relative velocity is

$$(8)\qquad \begin{cases} x^2 + y^2 + \dfrac{2(1-\mu)}{r_1} + \dfrac{2\mu}{r_2} = C, \\ r_1 = \sqrt{(x-x_1)^2 + y^2 + z^2}, \\ r_2 = \sqrt{(x-x_2)^2 + y^2 + z^2}. \end{cases}$$

Since y and z occur only in the squares the surfaces defined by (8) are symmetrical with respect to the xy and xz-planes, and, when $\mu = \frac{1}{2}$, with respect to the yz-plane also. The surfaces for $\mu \neq \frac{1}{2}$ may be regarded as being deformations of those for $\mu = \frac{1}{2}$. It follows from the way in which z enters that a line parallel to the z-axis will pierce the surfaces in two (or no) real points. Moreover, the surfaces are contained within a cylinder whose axis is the z-axis and whose radius is $\sqrt{C}$, to which certain of the folds are asymptotic at $z^2 = \infty$; for, as z^2 increases the equation approaches as a limit

$$x^2 + y^2 = C.$$

118. Approximate Forms of the Surfaces. From the properties of the surfaces just given, and their intersections with the reference planes, a general idea of their form may be obtained. The

equation of the curves of intersection of the surfaces with the xy-plane is obtained by putting z equal to zero in the first of (8), and is

$$(9) \qquad x^2+y^2+\frac{2(1-\mu)}{\sqrt{(x-x_1)^2+y^2}}+\frac{2\mu}{\sqrt{(x-x_2)^2+y}}=C$$

For large values of x and y which satisfy this equation the third and fourth terms are relatively unimportant, and the equation may be written

$$x^2+y^2=C-\frac{2(1-\mu)}{\sqrt{(x-x_1)^2+y^2}}-\frac{2\mu}{\sqrt{(x-x_2)^2+y^2}}=C-\epsilon,$$

where ϵ is a small quantity This is the equation of a circle whose radius is $\sqrt{C-\epsilon}$, therefore, one branch of the curve in the xy-plane is an approximately circular oval within the asymptotic cylinder It is also to be noted that the larger C is, the larger are the values of x and y satisfying the equation, the smaller is ϵ, the more nearly circular is the curve, and the more nearly does it approach its asymptotic cylinder

For small values of x and y satisfying (9) the first and second terms are relatively unimportant, and the equation may be written

$$\frac{1-\mu}{r_1}+\frac{\mu}{r_2}=\frac{C}{2}-\frac{x^2+y^2}{2}=\frac{C}{2}-\epsilon$$

This is the equation of the *equipotential curves* for the two centers of force, $1-\mu$ and μ For large values of C they consist of closed ovals around each of the bodies $1-\mu$ and μ, for smaller values of C these ovals unite between the bodies forming a dumb-bell shaped figure in which the ends are of different size except when $\mu=\frac{1}{2}$, and for still smaller values of C the handle of the dumb-bell enlarges until the figure becomes an oval enclosing both of the bodies

From these considerations it follows that the approximate forms of the curves in which the surfaces intersect the xy-plane are as given in figure 30

The curves C_1, C_2, C_3, C_4, C_5 are in the order of decreasing values of the constant C

The equation of the curves of intersection of the surfaces and the xz-plane is obtained by putting y equal to zero in equation (8), and is

$$(10) \qquad x^2+\frac{2(1-\mu)}{\sqrt{(x-x_1)^2+z^2}}+\frac{2\mu}{\sqrt{(x-x_2)^2+z^2}}=C$$

For large values of x and z satisfying this equation the second and third terms are relatively unimportant, and it may be written

$$x^2=C-\epsilon,$$

which is the equation of a symmetrical pair of straight lines parallel to the z-axis. The larger C is, the larger is the value of x which, for a given value of z, satisfies the equation, and, therefore, the smaller is ϵ. Hence, the larger C the closer the lines are to the asymptotic cylinder.

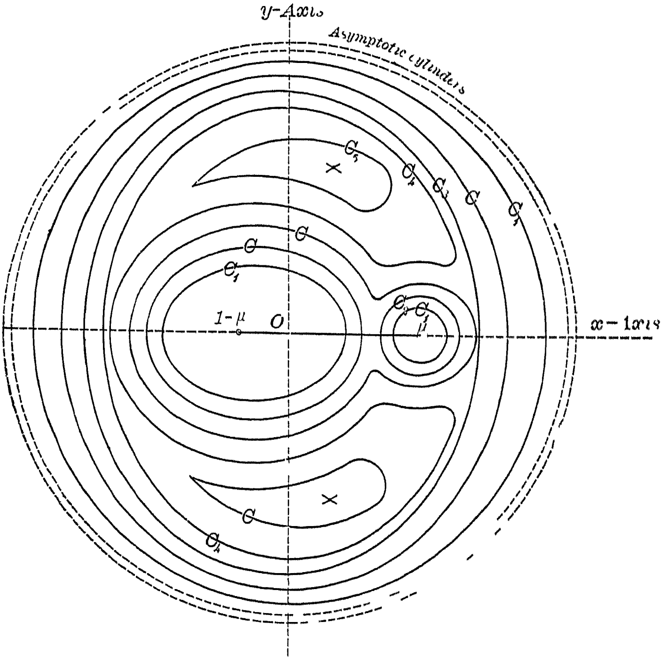

Fig. 30.

For small values of x and z satisfying equation (10) the first term is relatively unimportant, and the equation may be written

$$\frac{1-\mu}{r_1}+\frac{\mu}{r_2}=\frac{C}{2}-\epsilon.$$

This is again the equation of the equipotential curves and has the same properties as before. Hence, the approximate forms of the curves in the xz-plane are as given in figure 31.

Again, the curves $C_1, \ldots, C_5$ are in the order of decreasing values of the constant C.

The equation of the curves of intersection of the surfaces and the yz-plane is obtained by putting x equal to zero in equation (8) and is

$$(11) \qquad y^2+\frac{2(1-\mu)}{\sqrt{x_1^2+y^2+z^2}}+\frac{2\mu}{\sqrt{x_2^2+y^2+z^2}}=C.$$

For large values of y and z satisfying this equation the second and third terms are relatively unimportant, and it may be written

$$y^2 = C - \epsilon,$$

which is the equation of a pair of lines near the asymptotic cylinder, approaching it as C increases

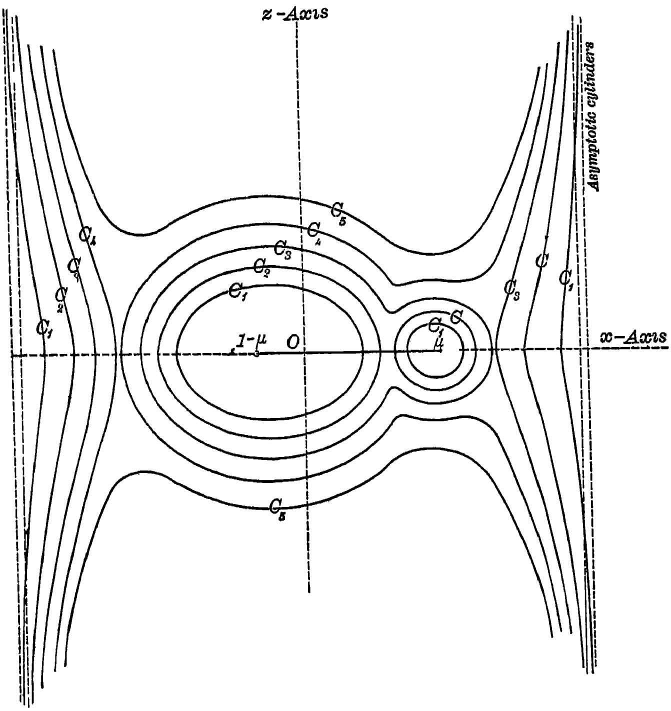

Fig 31

If $1 - \mu$ is much greater than μ, the numerical value of x_2 is much greater than that of x_1, hence, for small values of y and z satisfying (11), it may be written

$$\frac{2(1-\mu)}{r_1} = C - \epsilon,$$

which is the equation of a circle, which becomes larger as C decreases Hence, the approximate forms of the curves in the yz-plane are as given in figure 32

Again, the curves C_1, , C_5 are in the order of decreasing values of the constant C

From these three sections of the surfaces it is easy to infer their forms for the different values of C They may be roughly described as consisting of, for large values of C, a closed fold approximately spherical in form around each of the finite bodies, and of curtains hanging from the asymptotic cylinder symmetrically with respect to

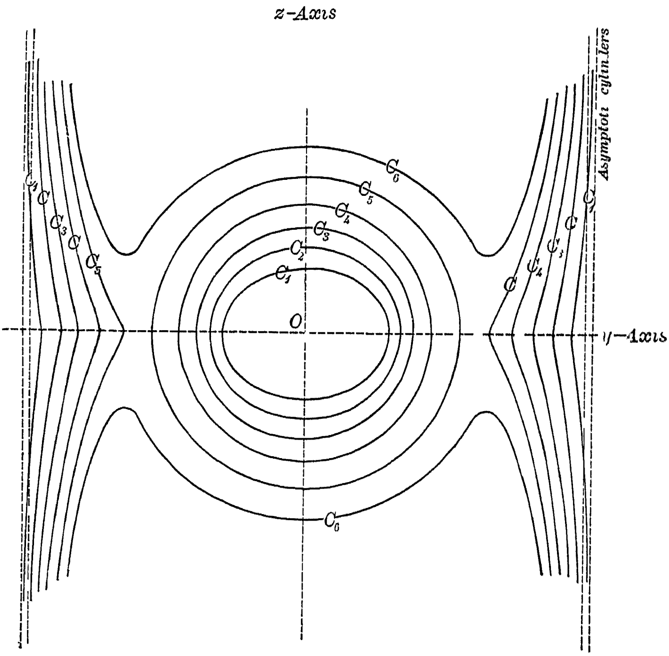

Fig 32

the xy-plane, for smaller values of C, the folds expand and coalesce, for still smaller values of C the united folds coalesce with the curtains, the first points of contact being in every case in the xy-plane, and for sufficiently small values of C the surfaces consist of two parts symmetrical with respect to the xy-plane but not intersecting it

119 The Regions of Real and Imaginary Velocity Having determined the forms of the surfaces, it remains to find in what regions of relative space the motion will be real and in what imaginary The equation for the square of the velocity is

$$V^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C$$

Suppose C is so large that the ovals and curtains are all separate. The motion will be real in those portions of relative space for which the right member of this equation is positive. If it is positive in one point in a closed fold it will be positive in every other point within it, for the function changes sign only at a surface of zero relative velocity.

It is evident from the equation that x and y may be taken large enough so that the right member will be positive, however great C may be; therefore, *the motion is real outside of the curtains*. It is also clear that a point may be chosen so near to either $1-\mu$ or μ, that is, either r_1 or r_2 may be taken so small, that the right member will be positive, however great C may be; therefore, *the motion is real within the folds around the finite bodies*.

If the value of C were so large that the folds around the finite bodies were closed, and if the infinitesimal body should be within one of these folds at the origin of time, it would always remain there since it could not cross a surface of zero velocity. If the earth's orbit be supposed to be circular and the mass of the moon infinitesimal, it is found that the constant C, determined by the motion of the moon, is so large that the fold around the earth is closed with the moon within it. Therefore the moon cannot recede indefinitely from the earth. It was in this manner that Hill proved that the moon's distance from the earth has a superior limit.*

120. Method of Computing the Surfaces. Actual points on the surfaces can be found most readily by first determining the curves in the xy-plane, and then finding by methods of approximation the values of z which will satisfy (7). Besides, the curves in the xy-plane are of most interest because the first points of contact as the various folds coalesce occur in this plane, and, indeed, on the x-axis, as can be seen from the symmetries of the surfaces.

The equation of the curves in the xy-plane is

$$x^2 + y^2 + \frac{2(1-\mu)}{\sqrt{(x-x_1)^2+y^2}} + \frac{2\mu}{\sqrt{(x-x_2)^2+y^2}} = C.$$

If this equation be rationalized and cleared of fractions the result is a polynomial of the sixteenth degree in x and y. When the value of one of the variables is taken arbitrarily the corresponding values of the other may be found by solving this rationalized equation. This problem presents great practical difficulties because of the high degree of the equation, and these troubles are supplemented by foreign solutions which are introduced by the processes of rationalization.

* *Lunar Theory*, *Am. Jour. Math.*, vol. I, p. 23.

The latter difficulty can be avoided entirely, and the degree of the equation very much reduced by transforming to bi polar coordinates That is, points on the curves may be defined by giving their distances from two fixed points on the x-axis This method could not be applied if the curves were not symmetrical with respect to the axis on which the poles lie Let the centers of the bodies $1-\mu$ and μ be taken as the poles, the distances from these points are r_1 and r_2 respectively To complete the transformation it is only necessary to express x^2+y^2 in terms of these quantities

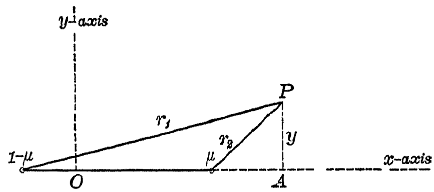

Fig 33

Let P be a point on one of the curves, then $OA=x$, $AP=y$, and, since O is the center of mass of $1-\mu$ and μ, $\overline{O\mu}=1-\mu$, and $\overline{O(1-\mu)}=-\mu$ It follows that

$$y^2=r_1^2-(x+\mu)^2=r_1^2-x-2\mu x-\mu,$$

$$y=r_2^2-\{x-(1-\mu)\}^2=r-x+2(1-\mu)x-(1-\mu)^2$$

Eliminating x from these equations, and solving for x^2+y^2, it is found that

$$x^2+y=(1-\mu)r_1^2+\mu r_2^2-\mu(1-\mu)$$

As a consequence of this equation, (9) becomes

$$(12)\qquad (1-\mu)\left(r_1+\frac{2}{r_1}\right)+\mu\left(r_2+\frac{2}{r_2}\right)=C+\mu(1-\mu)=C'$$

If an arbitrary value of r be assumed r_1 may be computed from this equation, the points of intersection of the circles around $1-\mu$ and μ as centers, with the computed and assumed values respectively of r_1 and r as radii, will be points on the curves To follow out this plan, equation (12) may be written

$$(13)\qquad \begin{cases} r_1^3+ar_1+b=0, \\ a=-\dfrac{C'}{1-\mu}+\dfrac{\mu}{1-\mu}\left(r_2+\dfrac{2}{r}\right), \\ b=2 \end{cases}$$

It follows from (12) that C' is always greater than $\mu\left(r_2^2+\dfrac{2}{r_2}\right)$ for all real positive values of r_1 and r_2, therefore a is always negative It

is shown in the Theory of Equations that the cubic equation of this form may be solved by means of trigonometrical functions as follows*

$$(14)\qquad \begin{cases} \sin\theta = \dfrac{b}{2}\sqrt{\dfrac{27}{-a^3}}, & \theta \leqq \dfrac{\pi}{2}, \\ r_{11} = 2\sqrt{\dfrac{-a}{3}}\sin\dfrac{\theta}{3}, & \\ r_{12} = 2\sqrt{\dfrac{-a}{3}}\sin\left(60^\circ - \dfrac{\theta}{3}\right), & \\ r_{13} = -2\sqrt{\dfrac{-a}{3}}\sin\left(60^\circ + \dfrac{\theta}{3}\right), & \end{cases}$$

where r_{11}, r_{12}, r_{13} are the three roots of the cubic

The only values of r_1 and r_2 having a meaning in this problem are real and positive It follows from the definition of θ that the three roots are real only if

$$-4a^3 \geqq 27b^2,$$

or, since $b=2$, if

$$(15)\qquad a+3 \leqq 0$$

The root r_{13} is negative and need not be considered The limit of the inequality (15) is $a+3=0$, or, in terms of the original quantities,

$$(16)\qquad \begin{cases} r_2^3 + a'r_2 + b' = 0, \\ a' = -\dfrac{C'}{\mu} + \dfrac{3(1-\mu)}{\mu}, \\ b' = 2 \end{cases}$$

The solution of this equation gives the extreme values of r_2 for which (13) has real roots Therefore, in the actual computation equation (16) should first be solved for r_{21} and r_{22} The values of r_2 to be substituted in (13) should be chosen at convenient intervals between these roots Equation (16) will not have real roots for all values of a', the condition for real roots being

$$a' + 3 \leqq 0,$$

the limiting value of which is in the original quantities,

$$-\frac{C'}{\mu} + \frac{3(1-\mu)}{\mu} = -3,$$

whence

$$C' = 3$$

Therefore C' must be equal to, or greater than, 3 in order that the curves may have real points in the xy-plane For $C'=3$ the curves are just vanishing from the plane, and it follows at once that equation

* See Chauvenet's *Plane and Spherical Trigonometry*, p 100

(12) is then satisfied by $r_1 = 1$, $r_2 = 1$, that is, the surfaces vanish from the xy-plane at the points which form equilateral triangles with $1 - \mu$ and μ

121 Double Points of the Surfaces, and Particular Solutions of the Problem of Three Bodies It follows from the general forms of the surfaces that the double points which appear as C diminishes are all in the xy-plane Therefore it is sufficient in this discussion to consider the equation of the curves in the xy-plane There are three double points on the x-axis which appear when the ovals around the finite bodies touch each other and when they touch the exterior curve enclosing them both There are two more which appear, as the surfaces vanish from the xy-plane, at the two points making equilateral triangles with the finite bodies

These double points are of interest as critical points of the curves, and it will now be shown that they are connected with important dynamical properties of the system Let the equation of the curves be written

$$(17) \qquad F(x, y) \equiv x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C = 0$$

The conditions for double points are

$$(18) \qquad \begin{cases} \dfrac{1}{2}\dfrac{\partial F}{\partial x} \equiv x - (1-\mu)\dfrac{(x - x_1)}{r_1^3} - \mu\dfrac{(x - x_2)}{r_2^3} = 0, \\ \dfrac{1}{2}\dfrac{\partial F}{\partial y} \equiv y - (1-\mu)\dfrac{y}{r_1^3} - \mu\dfrac{y}{r_2^3} = 0, \\ \quad z = 0 \end{cases}$$

The right members of these equations are the same as the right members of the differential equations (4) The expressions $\frac{1}{2}\frac{\partial F}{\partial x}$ and $\frac{1}{2}\frac{\partial F}{\partial y}$ are proportional to the direction cosines of the normal at all ordinary points of the curves, and since $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero at the surfaces of zero velocity it follows from (4) that *the directions of acceleration, or the lines of effective force, are orthogonal to the surfaces of zero relative velocity* Therefore, if the infinitesimal body be placed on a surface of zero relative velocity it will start in its motion in the direction of the normal But at the double points the sense of the normal becomes ambiguous, hence, it might be surmised that if the infinitesimal body were placed at one of these points it would remain relatively at rest

The conditions imposed by (17) and (18) are also the conditions that

$\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$, or the components of acceleration, in equations (4) shall vanish. Hence, *if the infinitesimal body be placed at a double point with zero relative velocity its coordinates will identically fulfill the differential equations of motion and it will remain forever relatively at rest, unless disturbed by forces exterior to the system under consideration.* These are particular solutions of the Problem of Three Bodies, and are special cases of the Lagrangian solutions.

Consider the equations (18), the second of which is satisfied by $y=0$. The double points on the x-axis, and the straight line solutions of the problem are given by the conditions

$$(19)\qquad \begin{cases} x-(1-\mu)\dfrac{(x-x_1)}{[(x-x_1)^2]^{\frac{3}{2}}}-\mu\dfrac{(x-x_2)}{[(x-x_2)^2]^{\frac{3}{2}}}=0, \\ y=0, \\ z=0. \end{cases}$$

The left member of the first equation considered as a function of x is positive for $x=+\infty$, it is negative for $x=x_2+\epsilon$, where ϵ is a very small positive quantity; it is positive for $x=x_2-\epsilon$, it is negative for $x=x_1+\epsilon$, it is positive for $x=x_1-\epsilon$, and it is negative for $x=-\infty$. Since the function is finite except when $x=+\infty$, x_2, x_1, or $-\infty$, it follows that the function changes sign three times by passing through zero, (*a*) once between $+\infty$ and x_2, (*b*) once between x_2 and x_1, and (*c*) once between x_1 and $-\infty$. Therefore, there are three positions in the line defined by $1-\mu$ and μ at which the infinitesimal body will remain when given proper initial projection.

(*a*) At the double point on the x-axis between $+\infty$ and x_2, $x-x_2=r_2$, $x-x_1=r_1=1+r_2$, $x=1-\mu+r_2$; therefore the first equation of (19) becomes after clearing of fractions

$$(20)\qquad r_2^5+(3-\mu)r_2^4+(3-2\mu)r_2^3-\mu r_2^2-2\mu r_2-\mu=0.$$

This quintic equation has one variation in the sign of its coefficients, and hence only one real positive root. The value of this root depends upon μ. Consider the left member of the equation as a function of r_2 and μ. For $\mu=0$ the equation becomes

$$r_2^3(r_2^2+3r_2+3)=0,$$

which has three roots $r_2=0$, and two others, coming from the second factor, which are complex. It follows from the theory of the solution of algebraic equations that, for μ different from zero but sufficiently small, three roots of the equation are expressible as power series in $\mu^{\frac{1}{3}}$, vanishing with this parameter.* The one of these three roots

* See Harkness and Morley's *Theory of Functions*, chapter IV.

obtained by taking the real value of $\mu^{\frac{1}{3}}$ is real, the other two are complex Therefore, the real root has the form

$$r_2 = a_1\mu^{\frac{1}{3}} + a_2\mu^{\frac{2}{3}} + a_3\mu^{\frac{3}{3}} + \cdots$$

Substituting this expression for r_2 in (20) and equating to zero the coefficients of the different powers of $\mu^{\frac{1}{3}}$, it is found that

$$a_1 = \frac{3^{\frac{2}{3}}}{3}, \quad a_2 = \frac{3^{\frac{1}{3}}}{9}, \quad a_3 = -\frac{1}{27},$$

Hence

$$(21) \qquad \begin{cases} r_2 = \mu^{\frac{1}{3}}\left(\dfrac{3^{\frac{2}{3}}}{3} + \dfrac{(3\mu)^{\frac{1}{3}}}{9} - \dfrac{\mu^{\frac{2}{3}}}{27} \cdots\right), \\ r_1 = 1 + r_2 \end{cases}$$

The corresponding value of C' is found by substituting these values of r_1 and r_2 in equation (12)

(*b*) At the double point on the x-axis between x_2 and x_1, $x - x_2 = -r_2$, $x - x_1 = r_1 = 1 - r_2$, $x = (1-\mu) - r_2$, therefore in this case the first equation of (19) becomes

$$r_2^5 - (3-\mu)r_2^4 + (3-2\mu)r_2^3 - \mu r_2^2 + 2\mu r_2 - \mu = 0$$

Solving as in (*a*), the values of r_2 and r_1 are found to be

$$(22) \qquad \begin{cases} r_2 = \mu^{\frac{1}{3}}\left(\dfrac{3^{\frac{2}{3}}}{3} - \dfrac{(3\mu)^{\frac{1}{3}}}{9} - \dfrac{\mu^{\frac{2}{3}}}{27} \cdots\right), \\ r_1 = 1 - r_2 \end{cases}$$

The corresponding value of C' is found by substituting these values of r_1 and r_2 in equation (12)

(*c*) At the double point on the x-axis between x_1 and $-\infty$, $x - x_2 = -(1 + r_1)$, $x - x_1 = -r_1$, $x = -\mu - r_1$, therefore in this case the first equation of (19) becomes

$$r_1^5 + (2+\mu)r_1^4 + (1+2\mu)r_1^3 - (1-\mu)r_1^2 - 2(1-\mu)r_1 - (1-\mu) = 0$$

For small values of μ, r_1 is nearly equal to unity, becoming exactly unity when μ vanishes, therefore it will be convenient to let $r_1 = 1 - \rho$, after which this equation becomes

$$(23) \qquad \rho^5 - (7+\mu)\rho^4 + (19+6\mu)\rho^3 - (24+13\mu)\rho^2 + (12+14\mu)\rho - 7\mu = 0$$

When $\mu = 0$ this equation becomes

$$\rho^5 - 7\rho^4 + 19\rho^3 - 24\rho^2 + 12\rho = 0,$$

which has but one root $\rho = 0$ Therefore ρ may be expressed as a power series in μ which converges for sufficiently small values of this parameter, and vanishes with it This root will have the form

$$\rho = c_1\mu + c_2\mu^2 + c_3\mu^3 + c_4\mu^4 + \cdots$$

Substituting this expression for ρ in (23), and equating to zero the coefficients of the various powers of μ, it is found that

$$c_1 = \frac{7}{12}, \quad c_2 = 0, \quad c_3 = \frac{23 \times 7^2}{12^4}, \quad c_4 = \frac{23 \times 7^3}{12^5}$$

Hence

$$(24) \qquad \begin{cases} \rho = \dfrac{7}{12}\mu + \dfrac{23 \times 7^2}{12^4}\mu^3 + \dfrac{23 \times 7^3}{12^5}\mu^4 + \quad , \\ r_1 = 1 - \rho, \\ r_2 = 1 + r_1 = 2 - \rho \end{cases}$$

The corresponding value of C' is found by substituting these values of r_1 and r_2 in equation (12)

If the values of r_1 and r given by the first three terms of the series (21), (22), and (24) are not sufficiently accurate more nearly correct values should be found by differential corrections

In order to find the double points *not* on the x-axis consider equations (18) again They, or any two independent functions of them, define the double points Since y is distinct from zero in this case the second equation may be divided by it, giving

$$1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} = 0$$

Multiply this equation by $x - x_2$, and $x - x_1$, and subtract the products separately from the first of (18) The results are

$$\begin{cases} x_2 - (1-\mu)\dfrac{(x_2 - x_1)}{r_1^3} = 0, \\ x_1 - \mu\dfrac{(x_1 - x_2)}{r_2^3} = 0, \\ z = 0 \end{cases}$$

But $x_2 = 1 - \mu$, $x_1 = -\mu$, and $x_2 - x_1 = 1$, therefore these equations reduce to

$$\begin{cases} 1 - \dfrac{1}{r_1^3} = 0, \\ -1 + \dfrac{1}{r_2^3} = 0, \\ z = 0 \end{cases}$$

The only real solutions are $r_1 = 1$, $r_2 = 1$, and the points form equilateral triangles with the finite bodies whatever their relative masses may be As was shown in the last of Art 120, they occur at the places where the surfaces vanish from the xy-plane

XIX PROBLEMS

1 The units defined in Art 115 are called *canonical units*, what would the canonical unit of time be in days for the earth and sun?

2 Show on *a priori* grounds that, when the motion of the system is referred to axes rotating as in Art 115, the differential equations should not involve the time explicitly

3 Why cannot an integral corresponding to (7) be derived from equations (1) at once without any transformations? Prove that there is an integral of (1)

4 What are the surfaces of zero velocity for a body projected vertically upward against gravity? For a body moving subject to a central force varying inversely as the square of the distance?

5 Show by direct reductions from (13) and (14) that

$$(r_1-r_{11})(r_1-r_{12})(r_1-r_{13})\equiv r_1^3+ar_1+b=0$$

6 Prove that the solution of (16) gives the extreme values of r_2 for which (14) has real roots

7 Impose the conditions on (12) that C' shall be a minimum and show that it is satisfied only for $r_1=1$, $r_2=1$, and that the minimum value of C' is 3

8 Why are not the lines of effective force orthogonal to all of the surfaces of constant velocity?

9 Prove that the double point between μ and $1-\mu$ is nearer μ than is the one between μ and $+\infty$

10 Prove that, as C' diminishes, the first double point to appear is the one between μ and $1-\mu$, the second, the one between μ and $+\infty$, the third, the one between $1-\mu$ and $-\infty$, and the last, that which makes an equilateral triangle with the finite bodies

11 If $\mu=\frac{1}{11}$, $1-\mu=\frac{10}{11}$, find the values of r_1, r_2, and C' from (21), (22), (24), and (12)

Ans

(21)	$r_2=0\ 340$,	$r_1=1\ 340$,	$C'=3\ 535$,
(22)	$r_2=0\ 276$,	$r_1=0\ 724$,	$C'=3\ 653$,
(24)	$r_2=1\ 947$,	$r_1=0\ 947$,	$C'=3\ 173$

12 From the approximate values of the last example find by the method of differential corrections more accurate values

Ans

(21)	$r_2=0\ 347$,	$r_1=1\ 347$,	$C'=3\ 534$,
(22)	$r_2=0\ 282$,	$r_1=0\ 718$,	$C'=3\ 653$,
(23)	$r_2=1\ 947$,	$r_1=0\ 947$,	$C'=3\ 173$

13 Considering the earth's orbit to be a circle, find the distance in miles from the earth to the double point which is opposite to the sun Would an infinitesimal body at this point be eclipsed?

Ans 930,240 miles

122. Tisserand's Criterion for the Identity of Comets* Comets sometimes pass near the planets in their revolutions around the sun, and then the elements of their orbits are greatly changed Jupiter because of his large mass and considerable distance from the sun, is especially potent in producing these perturbations which are so great in some cases as to change the orbit completely Since a comet has no characteristic features by which it may be recognized with certainty, its identity might be in question if it were not followed visually during the time of the perturbations

One way of testing the identity of two comets appearing at different epochs is to take the orbit of the earlier and compute the perturbations which it undergoes, and then compare the derived elements with those determined from the later observations, or, the start may be made with the elements of the later comet, and by inverse processes the earlier elements may be computed and the comparison made One or the other of these plans has been followed until recent years

But the question arises if there is not some relation among the elements which remains unaltered by the perturbations This is the question which Tisserand has answered in the affirmative in one of his characteristically elegant and important papers on Celestial Mechanics

Let the eccentricity of Jupiter's orbit be supposed equal to zero, and the mass of the comet infinitesimal While both of these assumptions are false they are very nearly fulfilled, and the error introduced will be inappreciable, especially as the comet will be near enough to Jupiter to suffer sensible disturbances only a very short time Under these suppositions, and when the units are properly chosen, the integral

$$(7) \qquad \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = x^2 + y^2 + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C$$

is true. This is an answer to the question, for, when the elements are known the velocity and coördinates may be computed at any time, and the motion referred to rotating axes by equations (2) Hence, to test the identity of two comets, compute the function (7) for each orbit and see if the constant C is the same for both If the two values of C are the same, the probability is very strong that only one comet has been observed, if they are different, the two comets are certainly distinct bodies

The process just explained has the inconvenience of involving considerable computation This may be largely avoided by expressing

* *Bulletin Astronomique*, vol VI p 289, and *Méc. Cél*, vol IV p 203

(7) in terms of the ordinary elements of the orbit. The first step is to express (7) in terms of coordinates measured from fixed axes. The equations of transformation are the inverse of equations (2), viz.,

$$\begin{cases} x = \xi \cos t + \eta \sin t, \\ y = -\xi \sin t + \eta \cos t, \\ z = \zeta. \end{cases}$$

From these equations it is found that

$$x^2 + y^2 = \xi^2 + \eta^2,$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 + \xi^2 + \eta^2 - 2\left(\xi \frac{d\eta}{dt} - \eta \frac{d\xi}{dt}\right).$$

Hence equation (7) becomes

$$(25) \qquad \left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 - 2\left(\xi \frac{d\eta}{dt} - \eta \frac{d\xi}{dt}\right) = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C.$$

Let r represent the distance of the comet from the origin, and i the angle between the plane of the instantaneous orbit and the $\xi\eta$-plane. Then equations (24), Art. 89, give

$$\left(\frac{d\xi}{dt}\right)^2 + \left(\frac{d\eta}{dt}\right)^2 + \left(\frac{d\zeta}{dt}\right)^2 = \frac{2}{r} - \frac{1}{a},$$

$$\left(\xi \frac{d\eta}{dt} - \eta \frac{d\xi}{dt}\right) = \sqrt{a(1-e^2)} \cos i.$$

Hence equation (25) becomes

$$(26) \qquad \frac{2}{r} - \frac{1}{a} - 2\sqrt{a(1-e^2)} \cos i = \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} - C.$$

In the case of Jupiter and the sun μ is less than one-thousandth. Therefore the origin is very near the center of the sun, and r_1 is sensibly equal to r. In both instances the elements will be determined when the comet is far from both Jupiter and the sun so that $\frac{2\mu}{r_1}$ and $\frac{2\mu}{r_2}$ will both be so small that they may be neglected; then (26) reduces to the simple expression

$$\frac{1}{a} + 2\sqrt{a(1-e^2)} \cos i = C.$$

It will be noticed that the elements of this formula are the instantaneous elements for motion around a unit mass situated at the center of mass of the finite bodies. The actual elements used in Astronomy are the elements referred to the center of the sun, with the sun as the attracting mass. Nevertheless, on account of the small

relative mass of Jupiter the two sets of elements are sensibly the same, and if the two orbits are of the same body, the equation

$$(27) \qquad \frac{1}{a_1} + 2\sqrt{a_1(1-e_1^2)}\cos i_1 = \frac{1}{a_2} + 2\sqrt{a_2(1-e_2^2)}\cos i_2$$

must be fulfilled, where the elements are those in actual use by astronomers Such is the criterion developed by Tisserand, and employed later by Schulhof and others

123 Stability of Particular Solutions Five particular solutions of the motion of the infinitesimal body have been found If the infinitesimal body be displaced a very little from the exact points of the solutions and given a small velocity it will either oscillate around these respective points, at least for a considerable time, or it will rapidly depart from them In the first case the particular solution from which the displacement is made is said to be *stable*, in the second case, it is said to be *unstable*

This must be formulated mathematically Consider the equations

$$(28) \qquad \begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = f(x, y), \\[2ex] \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = g(x, y) \end{cases}$$

Suppose $x = x_0$, $y = y_0$, where x_0 and y_0 are constants, is a particular solution of (28) That is,

$$f(x_0, y_0) = 0, \qquad g(x_0, y_0) = 0$$

Give the body a small displacement and a small velocity so that its coordinates and components of velocity are

$$(29) \qquad \begin{cases} x = x_0 + x', \\ y = y_0 + y', \\ \dfrac{dx}{dt} = \dfrac{dx'}{dt}, \\[2ex] \dfrac{dy}{dt} = \dfrac{dy'}{dt}, \end{cases}$$

where x', y', $\dfrac{dx'}{dt}$, and $\dfrac{dy'}{dt}$ are very small Making these substitutions in equations (28), they become

$$(30) \qquad \begin{cases} \dfrac{d^2x'}{dt^2} - 2\dfrac{dy'}{dt} = f(x_0 + x', y_0 + y'), \\[2ex] \dfrac{d^2y'}{dt} + 2\dfrac{dx'}{dt} = g(x_0 + x', y_0 + y') \end{cases}$$

Developing the right members by Taylor's formula, they become

$$f(x_0+x', y_0+y') = f(x_0, y_0) + \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y} y' + \text{higher powers in } x' \text{ and } y',$$

$$g(x_0+x', y_0+y') = g(x_0, y_0) + \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y} y' + \text{higher powers in } x' \text{ and } y'$$

In the partial derivatives $x = x_0$ and $y = y_0$ The first terms in the right members are respectively zero, hence equations (30) may be written

$$(31) \qquad \begin{cases} \dfrac{d^2x'}{dt^2} - 2\dfrac{dy'}{dt} = \dfrac{\partial f}{\partial x'} x' + \dfrac{\partial f}{\partial y'} y' + \quad, \\ \dfrac{d^2y'}{dt^2} + 2\dfrac{dx'}{dt} = \dfrac{\partial g}{\partial x'} x' + \dfrac{\partial g}{\partial y'} y' + \end{cases}$$

If x' and y' are taken very small on the start the influence of the higher powers in the right members will be inappreciable, at least for a considerable time Neglecting them, the differential equations reduce to the linear system

$$(32) \qquad \begin{cases} \dfrac{d^2x'}{dt^2} - 2\dfrac{dy'}{dt} = \dfrac{\partial f}{\partial x'} x' + \dfrac{\partial f}{\partial y'} y', \\ \dfrac{d^2y'}{dt^2} + 2\dfrac{dx'}{dt} = \dfrac{\partial g}{\partial x'} x' + \dfrac{\partial g}{\partial y'} y' \end{cases}$$

The solutions of a linear system with constant coefficients may be expressed in terms of exponentials in the form

$$x' = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t},$$
$$y' = \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t},$$

where α_1 and α_2 are the constants of integration, and β_1 and β are constants depending upon them and the constants involved in the differential equations If λ_1 and λ_2 are pure imaginary numbers, then x' and y' are periodic, and the solution from which the start was made is said to be *stable*, if λ_1 and λ_2 are real or complex numbers, then x' and y' change indefinitely with t, and the solution is said to be *unstable*

124 Application of the Criterion for Stability to the Straight Line Solutions The definitions and general methods of the last article will now be applied to the special cases which have arisen in the discussion of the motion of the infinitesimal body The original differential equations were (Art 115)

$$\begin{cases} \dfrac{d^2x}{dt^2} - 2\dfrac{dy}{dt} = x - (1-\mu)\dfrac{(x-x_1)}{r_1^3} - \mu\dfrac{(x-x_2)}{r_2^3} \equiv f(x, y, z), \\ \dfrac{d^2y}{dt^2} + 2\dfrac{dx}{dt} = y - (1-\mu)\dfrac{y}{r_1^3} - \mu\dfrac{y}{r_2^3} \equiv g(x, y, z), \\ \dfrac{d^2z}{dt^2} \qquad = -(1-\mu)\dfrac{z}{r_1^3} - \mu\dfrac{z}{r_2^3} \equiv h(x, y, z) \end{cases}$$

The straight line solutions occur for

$$x = x_{0i}, \quad y = 0, \quad z = 0,$$

where $i = 1, 2, 3$ according as the point lies between $+\infty$ and μ, μ and $1-\mu$, or $1-\mu$ and $-\infty$, and where these values of x, y, and z satisfy equation (19) Make the substitution

$$x = x_{0i} + x', \; y = y', \; z = z', \; \frac{dx}{dt} = \frac{dx'}{dt}, \; \frac{dy}{dt} = \frac{dy'}{dt}, \; \frac{dz}{dt} = \frac{dz'}{dt}$$

Then it is found that

$$(33) \quad \begin{cases} \dfrac{\partial f}{\partial x'}x' + \dfrac{\partial f}{\partial y'}y' + \dfrac{\partial f}{\partial z'}z' \equiv x' + \dfrac{2(1-\mu)x'}{[(x_{0i}-x_1)^2]^{\frac{3}{2}}} + \dfrac{2\mu x'}{[(x_{0i}-x_2)^2]^{\frac{3}{2}}}, \\ \dfrac{\partial g}{\partial x'}x' + \dfrac{\partial g}{\partial y'}y' + \dfrac{\partial g}{\partial z'}z' \equiv y' - \dfrac{(1-\mu)y'}{[(x_{0i}-x_1)^2]^{\frac{3}{2}}} - \dfrac{\mu y'}{[(x_{0i}-x_2)^2]^{\frac{3}{2}}}, \\ \dfrac{\partial h}{\partial x'}x' + \dfrac{\partial h}{\partial y'}y' + \dfrac{\partial h}{\partial z'}z' \equiv \quad \dfrac{-(1-\mu)z'}{[(x_{0i}-x_1)^2]^{\frac{3}{2}}} - \dfrac{\mu z'}{[(x_{0i}-x_2)^2]^{\frac{3}{2}}} \end{cases}$$

Let

$$(34) \qquad A_i = \frac{1-\mu}{[(x_{0i}-x_1)^2]^{\frac{3}{2}}} + \frac{\mu}{[(x_{0i}-x_2)^2]^{\frac{3}{2}}}$$

Then equations (32) become in this case

$$(35) \quad \begin{cases} \dfrac{d^2x'}{dt^2} - 2\dfrac{dy'}{dt} = (1+2A_i)x', \\ \dfrac{d^2y'}{dt^2} + 2\dfrac{dx'}{dt} = (1-A_i)y', \\ \dfrac{d^2z'}{dt^2} \qquad = -A_i z' \end{cases}$$

The last equation is independent of the first two and may be treated separately The solution is (Art 32)

$$(36) \qquad z' = c_1 e^{\sqrt{-1}\sqrt{A_i}t} + c_2 e^{-\sqrt{-1}\sqrt{A_i}t}$$

Therefore the motion parallel to the z-axis, for small displacements, is periodic with the period $\dfrac{2\pi}{\sqrt{A_i}}$

Consider now the simultaneous equations

$$(37)\qquad \begin{cases} \dfrac{d^2x'}{dt^2} - 2\dfrac{dy'}{dt} = (1 + 2A_i)\,x', \\[2ex] \dfrac{d^2y'}{dt^2} + 2\dfrac{dx'}{dt} = (1 - A_i)\,y' \end{cases}$$

To find the solutions let*

$$(38)\qquad \begin{cases} x' = Ke^{\lambda t}, \\ y' = Le^{\lambda t}, \end{cases}$$

where K and L are constants Substituting in equations (37) and dividing out $e^{\lambda t}$, it follows that

$$(39)\qquad \begin{cases} [\lambda - (1 + 2A_i)]\,K - 2\lambda L = 0, \\ 2\lambda K + [\lambda^2 - (1 - A_i)]\,L = 0 \end{cases}$$

In order that equations (38) may be particular solutions of (37) equations (39) must be fulfilled They are verified if $K = 0$, $L = 0$, but in this case $x' = 0$, $y' = 0$, and the solutions reduce to the straight line solutions Equations (39) may be satisfied by values of K and L different from zero if the determinant vanishes This condition is

$$(40)\qquad \begin{vmatrix} \lambda^2 - (1 + 2A_i), & -2\lambda \\ 2\lambda, & \lambda^2 - (1 - A_i) \end{vmatrix} \equiv \lambda^4 + (2 - A_i)\,\lambda + (1 + A_i - 2A_i^2) = 0$$

This equation is the condition upon λ that equations (38) may be a solution of (37) There are four roots of this biquadratic, each giving a particular solution, and the general solution is the sum of the four particular solutions multiplied by arbitrary constants, or, if the four roots of (40) are λ_1, λ , λ_3, λ_4,

$$(41)\qquad \begin{cases} x' = K_1e^{\lambda_1 t} + K_2e^{\lambda_2 t} + K_3e^{\lambda_3 t} + K_4e^{\lambda_4 t}, \\ y' = L_1e^{\lambda_1 t} + L_2e^{\lambda_2 t} + L_3e^{\lambda_3 t} + L_4e^{\lambda_4 t}, \end{cases}$$

where the K_i are the arbitrary constants of integration, and the L_i are defined in terms of them respectively by either of the equations (39)

It remains to determine the character of the roots of the biquadratic (40) It follows from (34) and (21), (22), and (24) respectively that

$$(42)\qquad \begin{cases} A_1 \equiv \dfrac{1-\mu}{(1+r_2)^3} + \dfrac{\mu}{r_2^3} = 4 - 2\ 3^{\frac{2}{3}}\mu^{\frac{1}{3}}, \\[2ex] A_2 \equiv \dfrac{1-\mu}{(1-r_2)^3} + \dfrac{\mu}{r_2^3} = 4 + 2\ 3^{\frac{2}{3}}\mu^{\frac{1}{3}}, \\[2ex] A_3 \equiv \dfrac{1-\mu}{(1-\rho)^3} + \dfrac{\mu}{(2-\rho)^3} = 1 + \frac{7}{8}\mu \end{cases}$$

* See Jordan's *Cours d'Analyse*, vol III, chap II, part 2

It follows from (42) that, for small values of μ,

$$1 + A_i - 2A_i^2 < 0, \quad (i = 1, 2, 3),$$

and, indeed, this relation is true for values of μ up to the limit $\frac{1}{2}$, as can be verified easily Therefore the biquadratic has two real roots which are equal in numerical value and opposite in sign, and two conjugate pure imaginaries It follows from the definitions given that the motion is unstable If the infinitesimal body were displaced a very little from the points of solution it would in general rapidly depart to a comparatively great distance

125 Particular Values of the Constants of Integration The constants of integration will now be expressed in terms of the initial conditions, and it will be shown that the latter may be selected so that the motion will be periodic

Suppose λ_1 and λ_2 are the real roots, then $\lambda_1 = -\lambda_2$ The imaginary roots are

$$\begin{cases} \lambda_3 = \sqrt{-1}\,\sigma, \\ \lambda_4 = -\sqrt{-1}\,\sigma, \end{cases}$$

where σ is a real number The L_j are expressed in terms of the K_j by equations (39), and are

$$(43) \qquad L_j = \frac{[\lambda_j^2 - (1 + 2A_i)]}{2\lambda_j} K_j = c_j K_j, \quad (j = 1, 2, 3, 4), (i = 1, 2, 3)$$

Since the λ_j are equal in numerical value but opposite in sign in pairs, and the last two imaginary, it follows that

$$(44) \qquad \begin{cases} c_1 = -c_2, \\ c_3 = +\sqrt{-1}\,c, \\ c_4 = -\sqrt{-1}\,c, \end{cases}$$

where c is a real constant

Let x_0', y_0', $\frac{dx_0'}{dt}$, and $\frac{dy_0'}{dt}$ be the initial coordinates and components of velocity, then equations (41) give at $t = 0$

$$\begin{cases} x_0' = K_1 + K_2 + K_3 + K_4, \\ y_0' = c_1 (K_1 - K_2) + \sqrt{-1}\,c\,(K_3 - K_4), \\ \dfrac{dx_0'}{dt} = \lambda_1 (K_1 - K_2) + \sqrt{-1}\,\sigma\,(K_3 - K_4), \\ \dfrac{dy_0'}{dt} = c_1\lambda_1 (K_1 + K_2) - c\sigma\,(K_3 + K_4) \end{cases}$$

Solving these equations, the values of the constants of integration are found in terms of the initial coordinates and components of velocity

The values of x' and y increase in general without limit with the time, but if the initial conditions are such that $K_1 = K_2 = 0$ they become purely periodic This case will now be considered The initial coordinates, x_0, y_0', will determine K_3 and K_4, by means of which $\frac{dx_0}{dt}$ and $\frac{dy_0'}{dt}$ are defined Thus

$$\begin{cases} x_0 = K_3 + K_4, \\ y_0' = \sqrt{-1}\, c\,(K_3 - K_4)\,, \end{cases}$$

whence

$$\begin{cases} K_3 = \dfrac{x_0'}{2} - \dfrac{\sqrt{-1}}{2c}\, y_0\,, \\[2ex] K_4 = \dfrac{x_0}{2} + \dfrac{\sqrt{-1}}{2c}\, y_0 \end{cases}$$

Then equations (41) become

$$(45)\quad \begin{cases} x' = \dfrac{x_0'}{2}\left(e^{\sqrt{-1}\sigma t} + e^{-\sqrt{-1}\sigma t}\right) - \dfrac{\sqrt{-1}}{2c}\, y_0'\left(e^{\sqrt{-1}\sigma t} - e^{-\sqrt{-1}\sigma t}\right) \\[2ex] \quad = x_0' \cos \sigma t + \dfrac{y_0'}{c} \sin \sigma t, \\[2ex] y' = \sqrt{-1}\,\dfrac{c}{2}\, x_0'\left(e^{\sqrt{-1}\sigma t} - e^{-\sqrt{-1}\sigma t}\right) + \dfrac{y_0'}{2}\left(e^{\sqrt{-1}\sigma t} + e^{-\sqrt{-1}\sigma t}\right) \\[2ex] \quad = -c x_0' \sin \sigma t + y_0' \cos \sigma t \end{cases}$$

The equation of the orbit is found by eliminating t from these equations Solve for $\cos \sigma t$ and $\sin \sigma t$, then square and add, and the result, after dividing out common factors, is

$$(46)\qquad \frac{x'^2}{\dfrac{c^2 x_0'^2 + y_0'^2}{c^2}} + \frac{y'}{c\, x_0'^2 + y_0'^2} = 1$$

This is the equation of an ellipse with the major and minor axes lying along the coordinate axes, and with the center at the origin Since λ_3 is imaginary it follows from (43) and (44) that $c > 1$, therefore the major axis of the ellipse is parallel to the y-axis The eccentricity is given by

$$e = \frac{c - 1}{c}\,,$$

which, for large values of c, is very near unity The orbits have the remarkable property that their eccentricity is independent of the initial small displacements, depending only upon the distribution of the mass between the finite bodies, and upon the one of the three straight line solutions from which they spring

126 **Application to the Gegenschein** If the constants K_1 and K_2 are zero the infinitesimal body will revolve in an ellipse around the point of equilibrium If these constants are not zero but small in numerical value compared to K_3 and K_4 the motion will be nearly in an ellipse for a considerable time, but will eventually depart very far from it It would be possible to have any number of infinitesimal bodies revolving around the same point without disturbing each other

Consider the motion of the earth around the sun It is in a curve which is nearly a circle One of the straight line solution points is exactly opposite to the sun, and if a meteor should pass near it with initial conditions approximately such as have been defined in the last article it would make one or more circuits around this point before pursuing its journey If a very great number were swarming around this point at one time they would appear from the earth as a hazy patch of light with its center at the anti-sun, and elongated along the ecliptic This is the appearance of the gegenschein which was discovered independently by Brorsen, Backhouse, and Barnard in 1855, 1868, and 1875 respectively

The crucial question seems to be whether or not there are enough meteors with the approximate initial conditions to explain the observed phenomena, but no certain answer can be given However, it is certain that the meteors are exceedingly numerous, as many as 8,000,000 striking into the earth's atmosphere daily according to the late Professor H A Newton, and it is only reasonable to suppose that they cause the zodiacal light which is very bright compared to the gegenschein

The celebrated Swedish astronomer, Hugo Gyldèn, first advanced the above theory in the closing paragraph of a memoir in the *Bulletin Astronomique*, vol I, entitled, *Sur un Cas Particulier du Problème des Trois Corps**

* See also a paper by F R Moulton in *The Astronomical Journal*, No 483

127 Application of the Criterion for Stability to the Equilateral Triangular Solutions The particular solutions of the original differential equations are $r_1 = 1$, $r_2 = 1$ The equations corresponding to (33) are

$$\begin{cases} \frac{\partial f}{\partial x} x' + \frac{\partial f}{\partial y'} y' + \frac{\partial f}{\partial z'} z' = \frac{3}{4} x' + \frac{3\sqrt{3}}{4}(1-2\mu) y', \\ \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial y'} y' + \frac{\partial g}{\partial z'} z' = \frac{3\sqrt{3}}{4}(1-2\mu) x' + \frac{9}{4} y', \\ \frac{\partial h}{\partial x'} x' + \frac{\partial h}{\partial y'} y' + \frac{\partial h}{\partial z'} z' = -z', \end{cases}$$

and the differential equations up to terms of the second degree are

$$(47) \qquad \begin{cases} \frac{d^2x'}{dt^2} - 2\frac{dy'}{dt} = \frac{3}{4} x' + \frac{3\sqrt{3}}{4}(1-2\mu) y', \\ \frac{d\,y'}{dt} + 2\frac{dx'}{dt} = \frac{3\sqrt{3}}{4}(1-2\mu) x' + \frac{9}{4} y', \\ \frac{d\,z'}{dt} \qquad\qquad = -z' \end{cases}$$

The last equation is independent of the first two, and its solution is

$$z = c_1 e^{\sqrt{-1}t} + c_2 e^{-\sqrt{-1}t}$$

Therefore the motion parallel to the z-axis, for small displacements, is periodic with period 2π, the same as that of the revolution of the finite bodies

To find the solutions of the first two equations let

$$(48) \qquad \begin{cases} x' = Ke^{\lambda t}, \\ y' = Le^{\lambda t} \end{cases}$$

Substituting in (47) and dividing out common factors, it is found that

$$(49) \qquad \begin{cases} [\lambda^2 - \frac{3}{4}] K - \left[2\lambda + \frac{3\sqrt{3}}{4}(1-2\mu)\right] L = 0, \\ \left[2\lambda - \frac{3\sqrt{3}}{4}(1-2\mu)\right] K + [\lambda - \frac{9}{4}] L = 0 \end{cases}$$

In order that solutions may be obtained other than $x' = 0$, $y' = 0$ the determinant of these equations must vanish That is,

$$(50) \qquad \begin{vmatrix} \lambda^2 - \frac{3}{4}, & -2\lambda - \frac{3\sqrt{3}}{4}(1-2\mu) \\ 2\lambda - \frac{3\sqrt{3}}{4}(1-2\mu), & \lambda - \frac{9}{4} \end{vmatrix} \equiv \lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0$$

Let λ_1, λ_2, λ_3, λ_4 be the roots of this biquadratic Then the general solutions are

$$\begin{cases} x' = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + K_3 e^{\lambda_3 t} + K_4 e^{\lambda_4 t}, \\ y' = L_1 e^{\lambda_1 t} + L_2 e^{\lambda_2 t} + L_3 e^{\lambda_3 t} + L_4 e^{\lambda_4 t}, \end{cases}$$

where K_1, K_2, K_3, K_4 are the constants of integration, and L_1, L_2, L_3, L_4 are constants related to them by either of the equations (49)

It is found from (50) that

$$\begin{cases} \lambda_1 = -\lambda_2 = \sqrt{\dfrac{-1 + \sqrt{1 - 27\mu(1-\mu)}}{2}}, \\ \lambda_3 = -\lambda_4 = \sqrt{\dfrac{-1 - \sqrt{1 - 27\mu(1-\mu)}}{2}} \end{cases}$$

The number μ never exceeds 1/2, and if $1 - 27\mu(1-\mu) \geqq 0$ the roots are pure imaginaries in conjugate pairs, if this inequality is not fulfilled they are complex quantities

The inequality may be written

$$1 - 27\mu(1-\mu) = \epsilon,$$

where ϵ is a positive quantity whose limit is zero The solution of this equation is

$$(51) \qquad \mu = \tfrac{1}{2} \pm \sqrt{\frac{23 + 4\epsilon}{108}}$$

Since μ represents the mass which is less than one-half the negative sign must be taken At the limit $\epsilon = 0$, $\mu = 0385$ Therefore if $\mu < 0385$ the roots of (50) are pure imaginaries and the equilateral triangular solutions are stable, if $\mu > 0385$ the roots of (50) are complex and the equilateral triangular solutions are unstable

XX PROBLEMS

1 Suppose a comet approaching the sun in a parabola should be disturbed by Jupiter so that its orbit remains a parabola while its perihelion distance is doubled, what is the relation between the new inclination and the old?

Ans $$\cos i_2 = \frac{\sqrt{2}}{2} \cos i_1$$

2 Prove that if a comet's orbit, whose inclination to Jupiter's orbit is zero, is changed by the perturbations of Jupiter from a parabola to an ellipse the parameter of the orbit is necessarily decreased Investigate the changes in the parameters for changes in the major axes of the other species of conics.

3 Suppose a comet is moving in an ellipse in the plane of Jupiter's orbit, and that the perturbing action of Jupiter is inappreciable except for a short time when they are near each other Prove that if the perturbation of Jupiter has increased the eccentricity, the period has been increased or decreased according as product of the major semi axis and the square root of the parameter in the original ellipse is greater or less than unity when expressed in the canonical units

4 A particle placed midway between two equal fixed masses is in equilibrium Investigate the character of the equilibrium by the method of Art 123

5 Suppose $1-\mu$ and μ are the sun and earth respectively, find the period of oscillation parallel to the z axis for an infinitesimal body slightly displaced from the xy plane near the straight line solution point opposite to the sun with respect to the earth as an origin

Ans 183 304 mean solar days

6 In the same case, find the period of oscillation in the xy plane

Ans 139 6 mean solar days

7 Prove that in general for small values of μ the periods of oscillation both parallel to the z axis and in the xy plane, are longest for the point opposite to μ with respect to $1-\mu$ as origin, next longest for the point opposite to $1-\mu$ with respect to μ as origin, and shortest for the point between $1-\mu$ and μ

8 Find the eccentricity of the orbit in the xy plane opposite to the sun in the case of the sun and earth

9 The differential equations (35) admit the integral

$$\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2+\left(\frac{dz'}{dt}\right)^2=(1+2A_i)\,x'^2+(1-A_i)\,y'^2-A_i z'^2+C,$$

discuss the meaning of this integral after the manner of articles 117—122

10 What can be said regarding the independence of equations (39) after the condition has been imposed that the determinant shall vanish ?

11 If the explanation of the gegenschein given in Art 126 is true what should be its maximum parallax in celestial latitude for an observer in latitude 45° ?

Ans Roughly 15′ (Too small to be observed with certainty in such an indefinite object)

Case of Three Finite Bodies

128 Conditions for Circular Orbits The theorem of Lagrange that it is possible to start three finite bodies in such a manner that their orbits will be similar ellipses, all described in the same time, will be proved in this section It will be established first for the special case in which the orbits are circles It will be assumed that the three bodies are projected in the same plane Take the origin at their center of mass and the $\xi\eta$-plane as the plane of motion Then the differential equations of motion are (Art 106)

$$
(52)\qquad \begin{cases} \dfrac{d^2\xi_i}{dt^2} = \dfrac{1}{m_i}\dfrac{\partial U}{\partial \xi_i}, & (i=1,2,3), \\ \dfrac{d^2\eta_i}{dt^2} = \dfrac{1}{m_i}\dfrac{\partial U}{\partial \eta_i}, & \\ U = \dfrac{k^2 m_1 m_2}{r_{12}} + \dfrac{k^2 m_2 m_3}{r_{23}} + \dfrac{k^2 m_3 m_1}{r_{31}} & \end{cases}
$$

The motion of the system is referred to axes rotating with the uniform angular velocity n by the substitution

$$
(53)\qquad \begin{cases} \xi_i = x_i \cos(nt) - y_i \sin(nt), \quad (i=1,2,3), \\ \eta_i = x_i \sin(nt) + y_i \cos(nt) \end{cases}
$$

Making the substitution, and reducing as in Art 115, it is found that

$$
(54)\qquad \begin{cases} \dfrac{d^2x_i}{dt^2} - 2n\dfrac{dy_i}{dt} - n^2 x_i - \dfrac{1}{m_i}\dfrac{\partial U}{\partial x_i} = 0, \\ \dfrac{d^2y_i}{dt^2} + 2n\dfrac{dx_i}{dt} - n^2 y_i - \dfrac{1}{m_i}\dfrac{\partial U}{\partial y_i} = 0 \end{cases}
$$

If the bodies were moving in circles around the origin with the angular velocity n, their coordinates with respect to the rotating axes would be constants Since the first and second derivatives would be zero, equations (54) would become

$$
(55)\qquad \begin{cases} -n^2x_1 + k^2m_2\dfrac{(x_1-x_2)}{r_{12}^3} + k^2m_3\dfrac{(x_1-x_3)}{r_{13}^3} = 0, \\ -n^2x_2 + k^2m_1\dfrac{(x_2-x_1)}{r_{12}^3} + k^2m_3\dfrac{(x_2-x_3)}{r_{23}^3} = 0, \\ -n^2x_3 + k^2m_1\dfrac{(x_3-x_1)}{r_{13}^3} + k^2m_2\dfrac{(x_3-x_2)}{r_{23}^3} = 0, \\ -n^2y_1 + k^2m_2\dfrac{(y_1-y_2)}{r_{12}^3} + k^2m_3\dfrac{(y_1-y_3)}{r_{13}^3} = 0, \\ -n^2y_2 + k^2m_1\dfrac{(y_2-y_1)}{r_{12}^3} + k^2m_3\dfrac{(y_2-y_3)}{r_{23}^3} = 0, \\ -n^2y_3 + k^2m_1\dfrac{(y_3-y_1)}{r_{13}^3} + k^2m_2\dfrac{(y_3-y_2)}{r_{23}^3} = 0 \end{cases}
$$

And conversely, if the masses and initial projections should be such that these equations are fulfilled the bodies would move in circles around the origin with the uniform angular velocity n

Since the origin is at the center of mass

$$
(56) \qquad \begin{cases} m_1 x_1 + m_2 x_2 + m_3 x_3 = 0, \\ m_1 y_1 + m_2 y_2 + m_3 y_3 = 0 \end{cases}
$$

If the first equation of (55) be multiplied by m_1, the second by m_2, and the products added, the sum becomes, as a consequence of the first equation of (56), the third of (55) In a similar manner the last equation of (55) may be derived from the others in y and the last of (56) Therefore the third and sixth equations of (55) may be suppressed, and equations (56) used in place of them, giving a somewhat simpler system of equations

The units of time, space, and mass are so far arbitrary It is possible, without loss of generality, to select them so that $r_{1\,2} = 1$, and $k^2 = 1$ Then the necessary and sufficient conditions for the solutions of the sort considered are

$$
(57) \qquad \begin{cases} m_1 x_1 + m_2 x_2 + m_3 x_3 = 0, \\ -n^2 x_1 + m_2 (x_1 - x_2) + m_3 \dfrac{(x_1 - x_3)}{r_{1\,3}^3} = 0, \\ -n^2 x_2 + m_1 (x_2 - x_1) + m_3 \dfrac{(x_2 - x_3)}{r_{2\,3}^3} = 0, \\ m_1 y_1 + m_2 y_2 + m_3 y_3 = 0, \\ -n^2 y_1 + m_2 (y_1 - y_2) + m_3 \dfrac{(y_1 - y_3)}{r_{1\,3}^3} = 0, \\ -n^2 y_2 + m_1 (y_2 - y_1) + m_3 \dfrac{(y_2 - y_3)}{r_{2\,3}^3} = 0 \end{cases}
$$

129 Equilateral Triangular Solutions There is a solution of the problem for every set of real values of the variables satisfying equations (57) It is easy to show that the equations are fulfilled if the bodies lie at the vertices of an equilateral triangle Then $r_{1\,2} = r_{2\,3} = r_{1\,3} = 1$, and equations (57) become

$$
\begin{cases} m_1 x_1 + m_2 x_2 + m_3 x_3 = 0, \\ (m_2 + m_3 - n^2)\, x_1 - m_2 x_2 - m_3 x_3 = 0, \\ (m_1 + m_3 - n^2)\, x_2 - m_1 x_1 - m_3 x_3 = 0, \\ m_1 y_1 + m_2 y_2 + m_3 y_3 = 0, \\ (m_2 + m_3 - n^2)\, y_1 - m_2 y_2 - m_3 y_3 = 0, \\ (m_1 + m_3 - n^2)\, y_2 - m_1 y_1 - m_3 y_3 = 0 \end{cases}
$$

Letting $M = m_1 + m_2 + m_3$, and eliminating x_3 and y_3 by means of the first and third equations, it follows that

$$\begin{cases} (M - n^2)\, x_1 = 0, \\ (M - n^2)\, x_2 = 0, \\ (M - n^2)\, y_1 = 0, \\ (M - n^2)\, y \ = 0 \end{cases}$$

These equations are fulfilled if $x_1 = x_2 = y_1 = y_2 = 0$, but then two of the bodies are at the center of mass which contradicts the assumption that $r_{1\,2} = r_{2\,3} = r_{1\,3} = 1$ The equations are also satisfied if $n^2 = M$ Therefore, *the equilateral triangular configuration with proper initial components of velocity is a particular solution of the Problem of Three Bodies, and, if the units are such that the mutual distances and k^2 are unity, the square of the angular velocity of revolution is equal to the sum of the masses of the three bodies*

130 Straight Line Solutions The last three equations of (57) are fulfilled by $y_1 = y_2 = y_3 = 0$, or when the bodies are all on the x-axis Suppose they lie in the order m_1, m_2, m_3 from the negative end of the axis toward the positive Then $x_3 > x_2 > x_1$, $r_{1\,2} = x_2 - x_1 = 1$, and the first three equations of (57) become

$$(58) \qquad \begin{cases} m_1 x_1 + m_2 (1 + x_1) + m_3 x_3 = 0, \\ m_2 + \dfrac{m_3}{(x_3 - x_1)^2} + n^2 x_1 = 0, \\ - m_1 + \dfrac{m_3}{(x_3 - x_1 - 1)^2} + n^2 (1 + x_1) = 0 \end{cases}$$

Eliminating x_3 and n^2, it is found that

$$(59) \qquad m_2 + (m_1 + m_2)\, x_1 + \frac{m_3{}^3 (1 + x_1)}{(M x_1 + m_2)^2} - \frac{m_3{}^3 x_1}{(M x_1 + m_2 + m_3)^2} = 0$$

Clearing of fractions this gives a quintic equation in x_1 with all its coefficients positive Therefore there is no real positive root but there is at least one real negative root, and consequently at least one solution of the problem

Instead of adopting x_1 as the unknown, $x_3 - x_2$, which will be denoted by A, may be used x_1 must be expressed in terms of this new variable The relations among x_1, x_2, x_3, and A are

$$\begin{cases} m_1 x_1 + m_2 x_2 + m_3 x_3 = 0, \\ x_2 - x_1 = 1, \\ x_3 - x_2 = A, \end{cases}$$

whence

$$x_1 = -\frac{m_2 + m_3 + m_3 A}{M}$$

Substituting in (59), clearing of fractions, and dividing out common factors, the condition for the solutions becomes

$$(60)\quad (m_1 + m_2) A^5 + (3m_1 + 2m_2) A^4 + (3m_1 + m_2) A^3 - (m_2 + 3m_3) A^2 - (2m_2 + 3m_3) A - (m_2 + m_3) = 0$$

This is precisely Lagrange's quintic equation*, and has but one real positive root in A since the coefficients change sign but once The only A valid in the problem for the chosen order of the masses is positive, hence the solution of (60) is unique and defines the distribution of the bodies in the straight line solution of the Problem of Three Bodies It is evident that two more distinct straight line solutions will be obtained by cyclically permuting the order of the three bodies

131 Dynamical Properties of the Solutions Since the bodies revolve in circles with uniform angular velocity around the center of mass, the law of areas holds for every body separately, therefore *the resultant of all the forces acting upon each body is constantly directed toward the center of mass* (Art 48)

Let the distances of m_1, m_2, and m_3 from their center of mass be a_1, a_2, and a_3 respectively Then the centrifugal acceleration to which m_i is subject is $\alpha_i = \frac{V_i^2}{a_i}$, where V_i is the linear velocity But this may be written $\alpha_i = n^2 a_i$ The centripetal force exactly balances the centrifugal, therefore the acceleration toward the center of mass is

$$(61)\qquad \alpha_i = n^2 a_i ,$$

that is, *the accelerations of the various bodies toward their common center of mass are directly proportional to their respective distances from this point*

132 General Conic Section Solutions If the three bodies were placed in either of the two given configurations, it would be necessary to project them at right angles to the radii to their center of mass with definite velocities proportional to their distances from the center of mass in order that the circular orbits should result

* See Lagrange's *Collected Works*, vol VI p 277, and Tisserand's *Méc Cél*, vol I p 155

It will now be shown that, if the bodies are initially placed in either of the given configurations, and are projected in lines making the same angles with their respective radii to their center of mass, in the same sense with respect to revolution around their center of mass, and with *any velocities* proportional to the respective distances of the bodies from their center of mass, then they will always form a figure of the same shape but in general of varying size, and will describe conic sections with respect to the origin as a focus

Suppose the bodies form one of the two configurations They will be subject to the attractive forces whose resultants are directed to their center of mass and which are proportional to their respective distances from it Suppose, however, that at the start they are subject to no forces whatever, and that they are projected in lines making equal angles with their respective radii, in the same sense with respect to revolution around the origin, and with velocities proportional to their respective distances from the center of mass At the end of an arbitrary unit of time the radii to the different bodies will all have described similar triangles, and the system will be of the same shape as at first Suppose that at this moment instantaneous forces act upon the bodies in the direction of their center of mass, and that the accelerations which they impart are proportional to the respective distances of the various bodies from that point Then, at the end of the next unit of time, the radii to the different bodies will all have described similar triangles, and the system will again be of the same shape as at first Suppose new instantaneous forces, of the same nature as those of the first case, act at the end of the second unit of time Suppose the force acting upon any given body in this case is to that in the first case inversely as the square of the distance of the body from the origin at the two instants Suppose this process continues during any finite interval of time, and then let the units into which the interval is divided become shorter and shorter The system will always be of the same shape but in general of varying size. At the limit the bodies are subject to continuous forces directed toward their center of mass and which vary for the different bodies directly as their respective distances from this point, and for any one body inversely as the square of its distance

But the forces which have just been supposed to act instantaneously at various times, have had the same directions, and have been of the same relative intensities as those arising from the mutual attractions of the bodies At the limit the fictitious forces are precisely the same,

when the factor of proportionality is properly chosen, as those due to the mutual attractions of the bodies Hence the theorem

If the three bodies are placed in one of the configurations for circular solutions, and are projected in lines making equal angles with their respective radii from their center of mass, and with velocities proportional to their respective distances, then they will always form a figure of the same shape but of varying size, and they will move subject to forces whose resultant on each body is directed to the center of mass of the system, and the resultants will vary for the different bodies directly as their respective distances, and for any one body inversely as the square of its distance from the center of mass

It follows from the theorem that the bodies move as though they were subject to central forces at the center of mass of the system, varying inversely as the square of their respective distances Therefore the orbits are conic sections, with a focus of each at the center of mass (Art 62), and, since the configuration always has the same shape, the conic sections are similar The extension to general conic sections when one of the three bodies is infinitesimal is made precisely as when the masses are all finite

XXI PROBLEMS

1 The Lagrangian solutions may be defined as those in which the ratios of the mutual distances are constants Taking this as the hypothesis, prove that the law of areas holds for each body with respect to either of the other two, with respect to the center of mass of the system, and with respect to the center of mass of any two of the bodies

2 It follows from Problem 1 that, since the center of mass may be supposed to be a fixed point, the resultant of the forces acting on each body always passes through this fixed point Prove that the equilateral triangular configuration is the only one for which this is true except when the bodies lie in a straight line

3 Write the conditions that the accelerations to which the bodies are subject shall be directed toward their common center of mass and proportional to their respective distances

Ans Equations (55)

4 Suppose $m_1=m_2=m_3=1$, and that the bodies move according to the equilateral triangular solution Find the radius of the circle in which a particle would revolve around one of them in the period in which they revolve around their center of mass

Ans $R=3^{-\frac{1}{3}}$

5 Prove that the equilateral triangular circular solutions hold when the mutual attractions of the bodies vary as any power of the distance

6 Prove that when the force varies inversely as the fifth power one solution is that each of the bodies moves in a circle through their center of mass in such a way that the three bodies are always at the vertices of an equilateral triangle

7 Prove that if the three bodies are placed at rest in any one of the configurations admitting circular solutions, they will fall to their center of mass in the same time in straight lines

8 Find the distribution of mass among the three bodies for which the time of falling to their center of mass will be the least, the greatest

9 Prove that if any four masses are placed at the vertices of a regular tetrahedron, the resultant of all the forces acting on each body will pass through the center of mass of the four, and that the magnitudes of the accelerations are proportional to the respective distances of the bodies from their center of mass

10 Prove that there are no circular solutions in the Problem of Four Bodies in which the bodies do not all move in the same plane

11 Investigate the stability of the circular and straight line solutions of the Problem of Three Bodies when all of the masses are finite

HISTORICAL SKETCH AND BIBLIOGRAPHY

The first particular solutions of the Problem of Three Bodies were found by Lagrange in his prize memoir, *Essai sur le Problème des Trois Corps*, which was submitted to the Paris Academy in 1772, (*Coll Works*, vol VI, p 229, Tisserand's *Méc Cél* vol I, chap VIII) The solutions which he found are precisely those given in the last part of this chapter His method was to divide the problem into two parts, (*a*) the determination of the mutual distances of the bodies, (*b*) having solved (*a*), the determination of the plane of the triangle in space and the orientation of the triangle in the plane He

proved that if the part (a) were solved the part (b) could also be solved To solve (a) it was necessary to derive three differential equations involving the three mutual distances alone as dependent variables He found three equations, one of which was of the third order, and the remaining two of the second order each, making the whole problem of the *seventh* order Reducing the general problem of three bodies by the ten integrals leaves it of the *eighth* order, hence Lagrange's analysis reduced the problem by one unit He found that he could integrate the differential equations completely by assuming that the ratios of the mutual distances were constants The demonstration was repeated by Laplace in the *Mécanique Céleste*, vol v, p 310 In *l'Exposition du Système du Monde* he remarked that if the moon had been given to the earth by Providence to illuminate the night, as some have maintained, the end sought has been only imperfectly attained, for, if the moon were properly started in opposition to the sun it would always remain there relatively, and the whole earth would have either the full moon or the sun always in view The demonstration upon which he based his remark was made under the assumption that there was no disturbing force If there were disturbing forces the configuration would not be preserved unless the solution were stable, which it is not, as was proved by Liouville, *Journal de Mathématiques*, vol VII, 1845

A number of memoirs have appeared following more or less closely along the lines marked out by Lagrange Among them may be mentioned one by Radau in the *Bulletin Astronomique*, vol III, p 113, by Lindstedt in the *Annales de l'École Normale*, 3rd series, vol I, p 85, by Allegret in the *Journal de Mathématiques*, 1875, p 277, by Bour in the *Journal de l'École Polytechnique*, vol XXXVI, and by Mathieu in the *Journal de Mathématiques*, 1876, p 345

Jacobi, without a knowledge of the work of Lagrange, reduced the general Problem of Three Bodies to the seventh order in *Crelle's Journal*, 1843, p 115, (*Coll Works*, vol IV, p 478) It has never been reduced further

Concerning the solutions of the problem of more than three bodies in which the ratios of the mutual distances are constants a number of papers have appeared, among which are one by Lehmann-Filhés in the *Astronomische Nachrichten*, vol CXXVII, p 137, and by F R Moulton in *The Transactions of the American Mathematical Society*, vol I, p 17

No new periodic solutions were discovered after those of Lagrange until Hill developed his Lunar Theory, *The American Journal of Mathematics*, vol I These solutions are of immensely greater practical value than those of the Lagrangian type It should be stated, however, that they are not strictly periodic solutions of any actual case, because a small part of the perturbing action of the sun was neglected

The next important advance was made by Poincaré in a memoir in the *Bulletin Astronomique*, vol I, in which he proved that when the masses of two of the bodies are small compared to that of the third, there are an infinite

number of sets of initial conditions for which the motion is periodic. These ideas were elaborated and the results extended in a memoir crowned with the prize offered by King Oscar of Sweden. This memoir appeared in *Acta Mathematica*, vol. XIII. The methods employed by Poincaré are incomparably more profound and powerful than any previously used in Celestial Mechanics, and mark an epoch in the development of the science. The whole work has been recast and extended in many directions, and published in three volumes entitled, *Les Méthodes Nouvelles de la Mécanique Celeste*. It is written with admirable directness and clearness, and is given in sufficient detail to make so profound a work as easily read as possible.

A memoir on Periodic Orbits by Darwin appeared in *Acta Mathematica*, vol. XXI. The methods employed were in many respects similar to those developed by Hill, and the results are of particular interest as they are all illustrated by numerical examples. The investigations were in regard to the motion of an infinitesimal body in the plane of motion of two finite bodies describing circles around their center of gravity.

The memoirs mentioned might well be read in the following order: That of Lagrange (in Tisserand, vol. I, chap. VIII.) and his followers; that of Jacobi; Hill's *Researches on the Lunar Theory*; Darwin's memoir; and Poincaré's *Méthodes Nouvelles*.

CHAPTER VIII

PERTURBATIONS—GEOMETRICAL CONSIDERATIONS

133 Meaning of Perturbations It was shown in Chapter V that if two spherical bodies move under the influence of their mutual attractions each describes a conic section with respect to their center of mass as a focus, and that the path of each body with respect to the other is a conic The converse theorem is also true, that is, if the law of areas holds and if the orbit of one body is a conic with respect to the other as a focus, then the force varies inversely as the square of the distance If there is a resisting medium, or if either of the bodies is oblate, or if there is a third body attracting the two under consideration, or if there is any force acting upon the bodies other than that of the mutual attractions of the two spheres, their orbits will cease to be exact conic sections The differences between the coordinates and the components of velocity in the actual orbits and those which the bodies would have had if the motion had been undisturbed are the *perturbations* It is necessary to include the changes in the components of velocity as perturbations, for the paths described depend not only upon the relative positions of the bodies and the forces to which they are subject, but also upon the relative velocities with which they are moving

Several methods of computing perturbations have been devised depending upon somewhat different points of view which may be taken Of these the two following are the ones most frequently used

134 Variation of Coordinates The simplest conception of perturbations is that the coordinates are directly perturbed No attempt is made to get the equations of the curve described, but the differences between the coordinates in the elliptic theory and the corresponding ones in the perturbed orbit are computed by the appropriate devices

135 Variation of the Elements This method is variously called the Variation of the Elements, the Variation of Parameters, and the Variation of the Constants of Integration According to this conception, a body is always moving in a conic section, but in one which changes at each instant The variable conic is tangent to the actual orbit at every point of it, and further, if the body moved continually in any one of the tangent conics it would have the same velocity at the point of tangency which it has in the actual orbit at that point This conic is said to osculate with the actual orbit at the point of contact. The perturbations are the differences between the elements of the orbit on the start, and those of the osculating conic at any time An obvious advantage of this method is that the elements change very slowly, since in the cases which actually arise the perturbing forces are small

This conception of perturbations arises quite naturally in considering the factors which determine the elements of an orbit It was shown in Chapter V that the initial positions of the two bodies and the directions of projection determine the plane of the orbit, that the initial positions and the velocities of projection determine the length of the major axis, and that the initial conditions, including the direction of projection and the velocities, determine the eccentricity and the line of the apsides

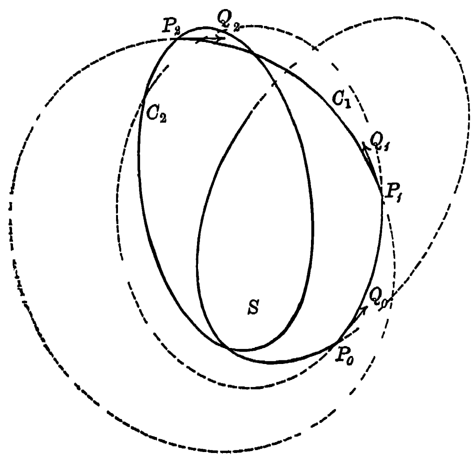

Fig 34

It was shown in Art 89 that the major semi axis is given by the very simple equation

$$a = \frac{r}{2}\left(\frac{U^2}{U^2 - V^2}\right), \tag{1}$$

where U^2 is the square of the velocity from infinity, and V^2 is the square of the velocity of projection Suppose the body m is projected from P_0 in the direction Q_0 with the velocity V_0 Suppose there are no forces acting upon it except the attraction of S, then, in accordance with what has been said above, it follows that it will move in a conic section whose elements are uniquely determined Suppose that when it arrives at P_1 it becomes subject to an instantaneous impulse of intensity F_1 in the direction $P_1 Q_1$, this position and the new velocity and direction of motion will determine a new conic C_1 in which the body will move until it is again disturbed by some external force Suppose it becomes subject to the impulse $P\ Q_2$ when it arrives at P, it will move in the new conic C_2 This may be supposed to continue indefinitely The body will be moving in conic sections which change at different intervals Suppose the instantaneous impulses become very small, and that the intervals of time between them become shorter and shorter The character of the motion will remain the same At the limit the impulses become a continually disturbing force, and the orbit a conic section which continually changes, hence the reasonableness of the conception

136 Derivation of the Elements from a Graphical Construction Suppose the major semi-axis has been computed by (1), it will be shown how the remaining elements may be found by the aid of very simple geometrical constructions The initial positions of S and m, and the direction of projection of m, determine the position of the plane of the orbit, and therefore Ω and i

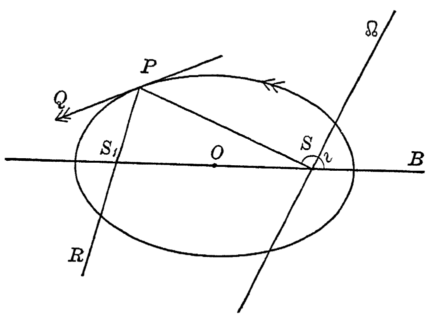

Fig 35

Suppose m is at the point P at the origin of time, and that it is projected in the direction PQ with the velocity V S is at one of the foci It is known from the properties of conic sections that the lines

from P to the two foci make equal angles with the tangent PQ Draw the line PR making the same angle with the tangent that SP makes Let r_1 represent the distance from S to P, and r_2 the distance from P to the second focus Therefore $r_1 + r_2 = 2a$, or, $r_2 = 2a - r_1$, which defines the position of S_1 Call the mid-point of SS_1, O, then $e = \frac{SO}{a}$ Suppose $S☊$ is the line of nodes, then the angle $☊SB = \omega$, and $\pi = \omega + ☊$

The only element remaining to be found is the time of perihelion passage The angle BSP, counted in the direction of motion, is v The eccentric anomaly is given by the equation (Art 98)

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{v}{2} \tag{2}$$

After E has been found the time of perihelion passage, T, is defined by the equation (Art 93)

$$n(t - T) = E - e\sin E \tag{3}$$

137 Resolution of the Disturbing Force Whatever be the source of the disturbing force it will be convenient, in order to find its effects upon the elements, to resolve it into three rectangular components It is possible to do this in several ways, each having advantages for particular purposes The one will be adopted here which leads most simply to the determination of the manner in which the elements vary when the body under consideration is subject to any disturbing force It would be possible without much difficulty to derive from geometrical considerations the expressions for the rates of change of the elements for any disturbing forces, but the object of this chapter is to explain the nature and causes of perturbations of various sorts, and the attention will not be divided by unnecessary digressions on methods of computation This part falls naturally to the methods of analysis, which will be given in the next chapter

The disturbing force will be resolved into three rectangular components (*a*) the *orthogonal component**, S, which is perpendicular to the plane of the orbit, and which will be taken positive when directed toward the plane of the ecliptic, (*b*) the *tangential component*, T, which is in the line of the tangent, and which will be taken positive when it acts in the direction of motion, and (*c*) the *normal component*, N, which is perpendicular to the tangent, and

* A designation due to Sir John Herschel, *Outlines of Astronomy*, p 420

which will be taken positive when directed to the interior of the ellipse

The instantaneous effects of these components upon the various elements will be discussed separately, and, unless it is otherwise stated, it always must be understood that the results refer to the way in which the elements are changing at given instants, and not to the cumulative effects of the disturbing forces Although the effects of the different components are considered separately, yet when two or more act simultaneously it is sometimes necessary to estimate somewhat carefully the magnitude of their separate perturbations, in order to determine the character of their joint effects

138 Disturbing Effects of the Orthogonal Component In order to fix the ideas and abbreviate the language it will be supposed that the disturbed body is the moon moving around the earth The perturbations arising from the disturbing action of the sun are very great and present many features of exceptional interest Besides, this is the case which Newton treated by methods essentially the same as those employed here* The character of the perturbations arising from positive components alone will be investigated, in every case negative components change the elements in the opposite way

It is at once evident that the orthogonal component will not change a, ω, T, and e, if ω is counted from a fixed line in the plane of the orbit But the ω in ordinary use is counted from the ascending node of the orbit, hence if the negative of the rate of increase of Ω be multiplied by $\cos i$ the result will be the rate of increase of ω due to the change in the origin from which it is reckoned Consequently it is sufficient to consider the changes in Ω and i when discussing the perturbations due to the orthogonal component

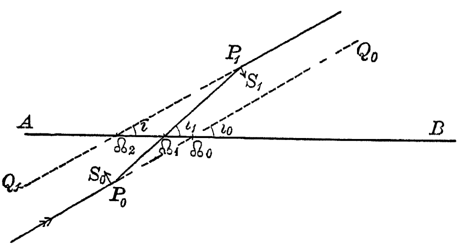

Fig 36

Let AB be in the plane of the ecliptic, $P_0 Q_0$ in the plane of the undisturbed orbit, and Ω_0 and i_0 the corresponding node and inclination

* *Principia*, Book I, Section 11, and Book III, Props XXII—XXXV

Suppose there is an instantaneous impulse $P_0 S_0$ when the moon is at P_0. It will then move in the direction $P_0 P_1$, and the new node and inclination will be ☊$_1$ and i_1. It is evident at once that $i_1 > i_0$, ☊$_1 <$ ☊$_0$. Suppose a new instantaneous impulse $P_1 S_1$ acts when the moon arrives at P_1. The new node and inclination are ☊$_2$ and i_2, and it is evident that $i_2 < i_1 > i_0$, ☊$_2 <$ ☊$_1 <$ ☊$_0$. If P_0☊$_1 =$ ☊$_1 P_1$, and $P_0 S_0 = P_1 S_1$, then, for small disturbances, $i_0 = i_2$. The total result is to cause the node to regress and the inclination to remain unchanged.

From the corresponding figure for the descending node it is seen that a positive S will produce the same result as at the ascending node. Therefore, *if the orthogonal component is positive and symmetrical with respect to the ecliptic during the whole revolution, the nodes will regress, but at an irregular rate, and the inclination, while varying irregularly, will retake its original value.*

139. Effects of the Tangential Component upon the Major Axis. Instead of deriving all the conclusions directly from geometrical constructions, it will be better to make use of some of the simple equations which have been found in Chapter V. If it were desired the theorems contained in these equations might be derived from geometrical considerations, as was done by Newton in the *Principia*, but this would involve considerable labor and would add nothing to the understanding of the subject.

The major semi-axis is given in terms of the initial distance and the initial velocity by the equation

$$V^2 = k^2 \left(\frac{2}{r} - \frac{1}{a}\right).$$

In an elliptic orbit a is positive; hence, since a positive T increases V^2, *a positive T increases the major semi-axes when the moon is in any part of its orbit.* It also follows from this equation that a given T is most effective in changing a when V^2 has its largest value, or when the moon is at the perigee.

140. Effects of the Tangential Component upon the Line of Apsides. The tangential component increases or decreases the speed, but does not change the direction of motion. The focus E is of course not changed, r_1 is unchanged, and, according to the results of the last article, a is increased. Since $r_2 = 2a - r_1$ while the direction of r_2 remains the same it follows that the focus E_1 will be thrown down to E_1'. The line of apsides will be rotated forward from

AB to $A'B'$ Hence it is easily seen that *a positive tangential component will cause the line of apsides to rotate forward during the first half revolution, and backward during the second half*

The instantaneous effects are the same for points which are symmetrical with respect to the major axis When the moon is at K or L the whole displacement of the second focus is perpendicular

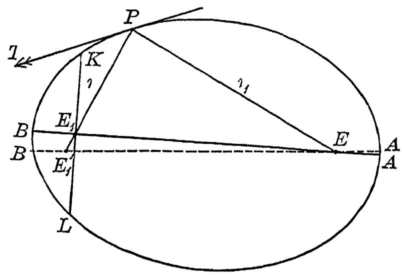

Fig 37

to the line of apsides, and at these points the rate of rotation of the apsides is a maximum for a given change in the major axis But the major axis is changed most when the moon is at perigee, therefore the place at which the line of the apsides will rotate most rapidly will be near K and L and between these points and the perigee The rate of rotation of the line of apsides becomes zero when the moon is at perigee and apogee

141 Effects of the Tangential Component upon the Eccentricity The eccentricity is given by the equation $e = \frac{EE_1}{2a}$ (Fig 37) When the moon is at the perigee EE_1 and $2a$ are increased by the same amount Since EE_1 is less than $2a$ the eccentricity is increased at this point When the moon is at apogee $2a$ is increased while EE_1 is decreased equally, hence the eccentricity is decreased Consequently there is some place between perigee and apogee where the eccentricity is not changed, and it is easy to show that this place is at the end of the minor axis Let $2\Delta a$ represent the instantaneous increase in $2a$ when the moon is at C or D Then r_2 will be increased by the quantity $2\Delta a$, and EE_1 by ΔE If θ is the angle CE_1E, $\cos\theta = \frac{EE_1}{2a} = \frac{2ae}{2a} = e$, and, moreover, $\Delta E = 2\Delta a \cos\theta = 2e\Delta a$ Therefore

$$(4) \qquad e' = \frac{EE_1 + \Delta E}{2a + 2\Delta a} = \frac{2ae + 2e\Delta a}{2a + 2\Delta a} = e,$$

or, the eccentricity is unchanged by the tangential component when the moon is at an end of the minor axis of its orbit

The changes in the time of perihelion passage depend upon the changes in the period and the direction of the major axis Since the period depends upon the major axis alone, whose changes have been

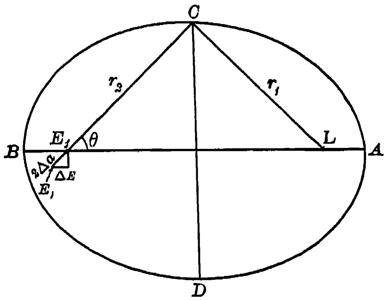

Fig 38

discussed, the foundations for an investigation of the changes in the time of perihelion passage have been laid, but further inquiry into this subject will be omitted as the results would have no applications in the work of this chapter

142 Effects of the Normal Component upon the Major Axis It follows from (1) that the major axis depends upon the speed at a given point and not upon the direction of motion Since the normal component acts at right angles to the tangent, it does not change the speed and, therefore, leaves the major axis unchanged

143 Effects of the Normal Component upon the Line of Apsides Consider the effect of an instantaneous normal component when the moon is at P Let PT represent the tangent to the orbit. The effect of the normal component will be to change it to PT' Since the radii to the two foci make equal angles with the tangent the radius r_2 will be changed to r_2', and, since the normal component does not affect the length of the major axis, r_2 and r_2' will be of equal length Consequently, *when the moon is in the region LAK a positive normal component will rotate the line of apsides forward, and when it is in the region KBL, backward* At the points K and L the normal component does not change the direction of the line of apsides

In the applications to the perturbations of the moon it will be necessary to determine the relative effectiveness of a given normal force in changing the line of apsides when the moon is at the two positions A and B When the moon is at either of these two points the second focus E_1 is displaced along the line KL The effectiveness

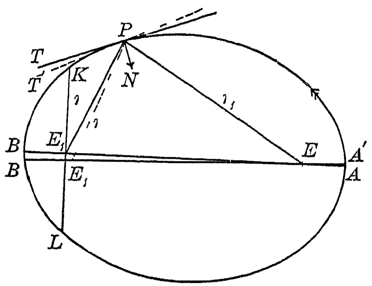

Fig 39

of a force in changing the direction of motion of a body is inversely proportional to the speed with which it moves, but the velocities at A and B are inversely proportional to their distances from E Let E_A and E_B represent the effectiveness of a given force in changing the direction of motion at A and B respectively, and let V_A and V_B represent the velocities at the same points Then

$$E_A \quad E_B = V_B \quad V_A = a(1-e) \quad a(1+e)$$

The rotation of the line of apsides is directly proportional to the displacement of E_1 along the line KL The displacements along KL are directly proportional to the products of the lengths of the radii from A and B to E_1 and the angles through which they are rotated But the angles are proportional to E_A and E_B, and the lengths of the radii to E_1 to $a(1+e)$ and $a(1-e)$ Therefore, letting R_A and R_B represent the rotation of the line of apsides at the two points, it follows that

$$R_A \quad R_B = E_A\, a(1+e) \quad E_B\, a(1-e) = 1 \quad 1,$$

or, *equal instantaneous normal forces produce equal, but oppositely directed, rotations of the line of apsides when the moon is at apogee and at perigee*

Suppose the forces act continuously over small arcs Since the linear velocities are inversely as the radii, *the effectiveness, in changing the direction of the line of apsides, of a constant force acting through a small arc at A is to that of an equal force acting through an equal arc at B as $a(1-e)$ is to $a(1+e)$* In practice the disturbing forces are

not instantaneous but act continuously, their magnitudes depending upon the positions of the bodies, consequently, unless the normal component is smaller at apogee than at perigee the average rotation of the line of apsides due to a normal component always having the same sign is in the direction of the rotation when the moon is at apogee

144 Effects of the Normal Component upon the Eccentricity If $2a$ represents the major axis, the eccentricity is given by (Fig 39)

$$e = \frac{EE_1}{2a}$$

After the action of the normal component the eccentricity is

$$e' = \frac{EE_1'}{2a},$$

the major axis being unchanged It is easily seen from the figure that *a positive normal force decreases the eccentricity during the first half revolution and increases it during the second half*, EE_1' being less than EE_1 in the first case and greater in the second The instantaneous change in the eccentricity vanishes when the moon is at A or B

145 Table of Results The various results obtained will be of constant use in the applications which follow, and they will be most convenient when condensed into a table The results are given for only positive values of the disturbing components, for negative components they are the opposite in every case

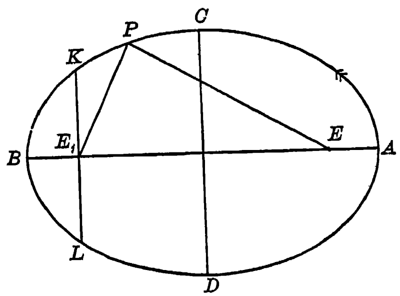

Fig 40

The orthogonal component, S, is positive when directed toward the ecliptic

The tangential component, T, is positive when directed in the direction of motion

The normal component, N, is positive when directed to the interior of the ellipse

Component	S	T	N
Node	Always retrogrades	0	0
Inclination	Increases before node passage, decreases after node passage	0	0
Major Axis	0	Always increases	0
Line of Apsides	0	In interval ACB, forward, In interval BDA, backward	In interval LAK, forward, In interval KBL, backward
Eccentricity	0	In interval DAC, increases, In interval CBD, decreases	In interval ACB, decreases, In interval BDA, increases

146 Disturbing Effects of a Resisting Medium The simplest disturbance of the elliptic motion is that arising from a resisting medium The only disturbing force is a negative tangential component, which has the same magnitude for points symmetrically situated with respect to the major axis Therefore, it is seen from the table that (1) ☊ and i are unchanged, (2) a is continually decreased, (3) the line of apsides undergoes periodic variations, rotating backward during the first half revolution, and rotating forward equally during the second half, (4) the eccentricity decreases while the body moves through the interval DAC, and increases during the remainder of the revolution It takes the body longer to move through the arc CBD than through DAC, but, on the other hand, if the resistance depends on a high power of the velocity, the change will be much greater at perigee than at apogee, and the whole effect in a revolution will be a decrease in the eccentricity

147 Perturbations Arising from Oblateness of the Central Body Consider the case of a satellite revolving around an oblate planet in the plane of its equator It was shown in Art 82 that the attraction under these circumstances is always greater than that of a concentric sphere of equal mass, but that the two attractions approach equality as the satellite recedes The excess of the attraction of the spheroid over that of an equal sphere will be considered as being the disturbing force, which, it will be observed, acts in the line of the radius vector and is always directed toward the planet Therefore the normal

component is always positive, and is equal in value at points which are symmetrically situated with respect to the major axis If the eccentricity of the orbit is not large the tangential component is relatively small, being negative in the interval ACB, and positive in BDA

(a) *Effect upon the period* This is most easily seen when the orbit is a circle The attraction will be constant and greater than it would be if the planet were a sphere This is equivalent to increasing k^2, the acceleration at unit distance, therefore it is seen from the equation

$$P = \frac{2\pi a^{\frac{3}{2}}}{k\sqrt{m_1 + m_2}}$$

that for a given orbit the period will be shorter, and for a given period the distance greater, than it would be if the planet were a sphere

(b) *Effects upon the elements* Referring to the table it is seen that (1) ☊ and i are unchanged, (2) a decreases and increases equally in a revolution, (3) the line of apsides rotates forward during a little more than half a revolution, and that while the disturbing force is of greatest intensity, and (4) the eccentricity is changed equally in opposite directions in a whole revolution That is, ☊ *and* i *are absolutely unchanged,* a *and* e *undergo periodic variations which complete their period in a revolution, and the line of apsides oscillates, but advances on the whole*

The effects will be the greater the more oblate the planet and the nearer the satellite The oblateness of the earth is so small that it has very little effect in rotating the moon's line of apsides The most striking example of perturbations of this sort in the solar system is in the orbit of the Fifth Satellite of Jupiter This planet is so oblate and the satellite's orbit is so small that its line of apsides progresses 900 in a year

XXII PROBLEMS

1 A body subject to no forces moves in a straight line with uniform speed The elements of this orbit are the constants which define the position of the line, viz the speed, the direction of motion in the line, and the position of the body at the time T Show that they may be expressed in terms of six independent constants, and that it is permissible in the problem of two bodies to regard one body as always moving with respect to the other in a straight line whose position continually changes Find the expression of these line elements in terms of the time in the case of elliptic motion

2 Show from general considerations that the methods of the variation of coordinates and the variation of parameters are essentially the same, differing only in the variables used in defining the coordinates and velocities of the bodies

3 Suppose the sun moves through space in the line L, orthogonal to the plane Π Take Π as the fundamental plane of reference Let the point where the planet P_i passes through the plane Π in the direction of the motion of the sun be the ascending node, and, beginning at this point divide the orbit into quadrants with respect to the sun as center Suppose the ether and scattered meteoric matter slightly retard the sun and the planets, but neglect the retardation arising from the motion of the planets in their orbits around the sun

(a) If the resistance is proportional to the masses of the respective bodies, show that the nodes and inclinations of their orbits are unchanged

(b) Let σ and R represent the density and radius of the sun, and σ_i and R_i the corresponding quantities for the planet P_i Then, if the resistance is proportional to the surfaces of the respective bodies, show that with respect to the plane Π the inclination and line of nodes undergo the following variations

(1) If $\sigma_i R_i < \sigma R$

Quadrant	1	2	3	4
Inclination	decreases	increases	increases	decreases
Line of nodes	regresses	regresses	advances	advances

(2) If $\sigma_i R_i > \sigma R$

Quadrant	1	2	3	4
Inclination	increases	decreases	decreases	increases
Line of nodes	advances	advances	regresses	regresses

(*c*) If the orbits were circles the various changes in both cases would exactly balance each other in a whole revolution How must the lines of apsides in the two cases lie with respect to the line of nodes in order that, for a few revolutions, (1) the inclination shall decrease the fastest, and (2) the line of nodes advance the fastest?

(*d*) Is it possible to make the relation of the line of apsides to the line of nodes such that, for a few revolutions, the inclination shall decrease and the line of nodes advance?

(*e*) If the line of apsides remains fixed in the plane of the orbit is it possible for the line of nodes to rotate indefinitely in one direction?

4 Suppose the orbit of a comet passes near Jupiter's orbit at one of its nodes, under what conditions will the inclination of the orbit of the comet be decreased? Show that if the major axis remains constant while the inclination is decreased the eccentricity is increased (Use Art 122)

5 What is the effect of the gradual accretion of meteoric matter by a planet upon the major axis of its orbit?

Ans It is gradually decreased

6 Consider two viscous bodies revolving around their common center of mass, and rotating in the same direction with periods less than their period of revolution They will generate tides in each other which will lag The tidal protuberances of each body will exert a positive tangential and a positive normal component on the other, these components being greater the nearer the bodies are together Moreover, the rotation of each body will be retarded by the action of the other on its protuberances Suppose the bodies are initially near each other and that their orbits are slightly elliptic, follow out the evolution of all of the elements of their orbits

148 Disturbing Effects of a Third Body This case is much more complicated than those heretofore treated, since the disturbing force varies in a very complicated fashion In order to find it the following lemma must be introduced *If two bodies are subject to equal parallel accelerations their relative motions are not changed* The method of demonstration is so evident that the proof will be omitted

Suppose the three bodies are S, E, and m Consider S as disturbing m in its motion around E Pass a plane through the three bodies Since the positions of the bodies are known the position of the

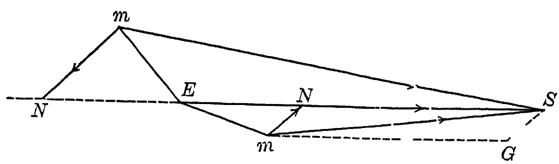

Fig 41

triangle with respect to any system of coordinates is known Let mS* represent in magnitude and direction the acceleration of S upon m † With the same units let NS represent in magnitude and direction the acceleration of S upon E (N and N' are at different distances because the units are different in the two cases) The relation between NS and mS is given by the equation

$$NS : mS = \overline{mS}^2 : \overline{ES}^2 ,$$

whence

$$NS = \left(\frac{mS}{ES}\right)^2 mS \tag{5}$$

Now resolve the acceleration mS into two components, so that one of them, mG, shall be equal and parallel to NS By the parallelogram of accelerations the other will be mN By the lemma mG and NS produce no disturbances in the relative motions of m and E, therefore mN, the remaining component of attraction on m, is the disturbing acceleration The other sides of the triangle mNS, and the angle NSm being known, mN can at once be expressed in terms of known quantities

* The order of the letters indicates the direction of the vector representing the force

† The statements are true for the letters with the accents, corresponding to another arrangement of the three bodies as well as for those without

149 Perturbations of the Node and Inclination The most obvious feature of the disturbing acceleration is that it is always directed so that it has a tendency to bring m in a line with ES If the plane of the orbit of S (which is now supposed to be undisturbed) be taken as the plane of reference, it follows that the orthogonal component is always directed toward it, or is zero It is zero when m is at one of the nodes of its orbit, for then the triangle EmS lies in the reference plane, it is also zero when m and E are equally distant from S, for then the entire disturbing acceleration is directed toward E, or is a radial component

(*a*) Since the orthogonal component is always positive (or zero) it follows from the results given in the table that the nodes will always regress (or remain stationary) upon the plane of the orbit of S, but at an irregular rate

(*b*) The inclination will sometimes increase and sometimes decrease, the rate of change depending upon the distance from the node and upon the magnitude of the disturbing acceleration, that is, upon the distance and direction of m from E and S In a single revolution the increase and the decrease of the inclination will in general not be equal because of a lack of symmetry in the conditions, but, if the periods of m and S are incommensurable, the inclination will not continue to change in one direction indefinitely After a certain lapse of time the relations of the positions of the bodies to the line of nodes will be such that the inclination will change on the average in the other direction for an equal time When the rotation of the line of nodes is included the same result follows, but with a different period

Therefore, *the line of nodes continually regresses, or is stationary, while the inclination undergoes short periodic variations, and another periodic variation of very long period which depends upon the periods of revolution and the rate of motion of the line of nodes*

150 Precession of the Equinoxes Nutation Suppose the largest sphere possible is cut out of the earth leaving an equatorial ring Every particle in this ring may be considered as being a small satellite, then, from the principles explained above, the attraction of the sun will exercise a disturbing acceleration upon it in the direction of the ecliptic, and the attraction of the moon a disturbing acceleration in the direction of the plane of its orbit The angle between these two planes may be neglected for the moment as it is very small compared to the inclination of the earth's equator The particles of the ring are fastened to the solid earth so that it partakes of any disturbance to which they may be

subject Since their combined mass is very small compared to that of the spherical body within them, and since the disturbing forces are very slight, the changes in the motion of the earth will take place very slowly

From the results of the last article it follows that the nodes of the orbit of every particle will have a tendency to regress on the plane of the disturbing body They communicate this tendency to the whole earth so that the plane of the earth's equator turns in the retrograde direction on the plane of the ecliptic On the other hand, it follows from the symmetry of the figure with respect to the nodes of the orbits of the particles of the equatorial ring, that there will be no change in the inclination of the plane of the equator to that of the ecliptic or the moon's orbit The mass moved is so great, and the forces acting are so small, that this retrograde motion, called the *precession of the equinoxes*, amounts to only about 50″ 2 annually, or, the plane of the earth's equator makes a revolution in about 26,000 years

Since the moon is so near to the earth compared to the sun, the orthogonal component arising from its attraction is greater than that coming from the sun's attraction The main regression is, therefore, on the moon's orbit which is inclined to the ecliptic about 5° 7′ Since the line of the moon's nodes makes a revolution in about 19 years, the plane with respect to which the equator regresses performs a revolution in the same time This produces a slight nodding in the motion of the pole of the equator around the pole of the ecliptic, and is called *nutation*

151 Resolution of the Disturbing Acceleration in the Plane of Motion An idea of the effects of the tangential and normal components can be most readily obtained by supposing that the mutual inclination of the two orbits is zero The method is the same in its general features in all cases, but, as the perturbations of the motion of the moon by the sun furnish some of the most interesting theorems in Celestial Mechanics, the treatment given here will be applied to the Lunar Theory in particular

Let E represent the earth, S the sun, and m the moon Let mS represent in magnitude and direction the acceleration of the sun upon the moon* Let NS represent in the same units the acceleration of S upon the earth It is given by the equation (Art 148)

$$(5) \qquad NS = \left(\frac{mS}{ES}\right)^2 mS$$

Then mN is the disturbing acceleration in magnitude and direction

* The statements are all true for the letters with accents which represent a different arrangement of the bodies, as well as for those without

The point N can be found approximately in a very simple manner in the Lunar Theory because of the great relative distance of the sun Let $Em = r$, $ES = R$, $mS = \rho$, and $EH = h$ Then it follows from Fig 42 and equation (5) that

$$EN = ES - NS = R - \frac{\rho^3}{R^2} = \frac{R^3 - \rho^3}{R^2} \tag{6}$$

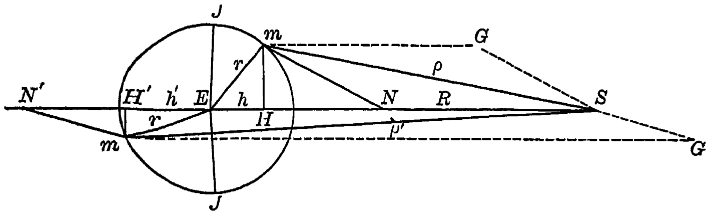

Fig 42

Since the angle mSE is always very small (less than 9) $R = h + \rho$ approximately Therefore (6) becomes

$$EN = \frac{h^3 + 3h^2\rho + 3h\rho^2 + \rho^3 - \rho^3}{h^2 + 2\rho h + \rho^2} \tag{7}$$

h is very small compared to ρ (less than $\frac{1}{400}$ of ρ), and all the terms in which it occurs except that of lowest degree may be neglected Therefore (7) becomes with close approximation

$$EN = \frac{3h\rho^2}{\rho^2} = 3h \tag{8}$$

From this it follows that the disturbing acceleration may be found approximately by taking the point N on the line passing through the sun and the earth such that its distance from the earth is three times the projection of the moon's radius vector from the earth on this line, then the line joining the moon and N will represent in magnitude and direction the disturbing acceleration The units are different when the moon is in different parts of her orbit, but the sun is so far from the earth relatively that this error generally need not be considered in purely qualitative work

When the moon is in conjunction with, or opposition to, the sun, the disturbing acceleration is, neglecting the eccentricity of the moon's orbit, all normal and negative, while at the points J, which are very nearly in exact quadrature, the disturbing acceleration is all normal and positive Moreover, it is twice as great in the former case as in the latter At some point between these points the normal component

is zero and the disturbing acceleration is all tangential This occurs when the angle EmN equals 90° At this point

$$h \quad mH = mH \quad HN = mH \quad 2h\,,$$

whence

$$\overline{mH}^2 = 2h^2,$$

and

$$(9) \qquad \tan mEH = \frac{mH}{h} = \frac{\sqrt{2}h}{h} = \sqrt{2}$$

Therefore

$$(10) \qquad \text{angle } mEH = 54\ \ 44'$$

The variations in sign of the normal and tangential components are indicated in the following figures

Normal Component

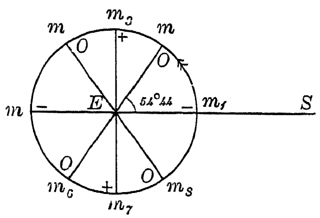

Fig 43

Tangential Component

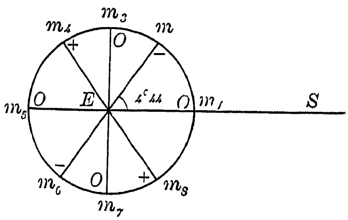

Fig 44

According to equation (8) the disturbing accelerations at points on the side of the orbit in opposition to the sun are apparently equal to those at the corresponding points on the side toward the sun, but they are actually somewhat smaller since for these points the units are different, a longer line representing a smaller acceleration

152 Perturbations of the Major Axis If the perigee were at m_1 or m_5 the T-component, which alone changes a, would be equal and of opposite sign at points symmetrically situated with respect to the major axis In this case a would be unchanged at the end of a complete revolution But this condition of affairs is only realized instantaneously, for the disturbing body S is moving in its orbit, yet, in a very large number of revolutions, when the periods are incommensurable, an equal number of equal positive and negative T-components will have exerted a disturbing influence The result will be that in the long run a is unchanged, although it undergoes periodic variations

153 Perturbation of the Period The normal component is not only negative more than half a revolution, but the negative values are greater numerically than the positive ones One effect of the whole result is equivalent to a diminution, on the average, of the attraction of E for m, that is, to a diminution of k^2, the acceleration at unit distance The relation of the period to the intensity of the attraction and the major axis is (Art 89)

$$P = \frac{2\pi a^{\frac{3}{2}}}{k\sqrt{E+m}}$$

Hence, for a given distance, P is increased if k is decreased. In this manner the sun's disturbing effect upon the orbit of the moon increases the length of the month by more than an hour (Compare Art 147 (a))

154 The Annual Equation The orbit of the earth being an ellipse the distance of the sun undergoes considerable variations The farther the sun is from the earth the feebler are its disturbing effects, and in particular, the power of lengthening the month considered in the last article Therefore, as the earth moves from perihelion to aphelion the disturbance which *increases* the length of the month will become less and less, that is, the length of the month will become shorter, or the moon's angular motion will be accelerated While the earth is moving from aphelion to perihelion the moon's motion will, for the opposite reason, be retarded This is the *Annual Equation* amounting to a little more than 11′, and was discovered from observations by Tycho Brahe about 1590

155 The Secular Acceleration of the Moon's Mean Motion In the early part of the 18th century Halley found from a comparison of ancient and modern eclipses that the mean motion of the moon is gradually increasing Nearly 100 years later (1787) Laplace gave the explanation of it, showing that it is caused by the gradual average decrease of the eccentricity of the earth's orbit, which has been going on for many thousands of years, and which will continue for a long time yet before it begins to increase

One effect of a change in the eccentricity of the earth's orbit is to change the average disturbing power of the sun on the orbit of the moon, it will now be shown that if the eccentricity decreases, the average disturbing power decreases

The disturbing effect of the sun upon the motion of the moon is

the difference of its acceleration upon the earth and moon When the three bodies are in a line the disturbing component is

$$(11) \qquad D = k^2 S\left[\frac{1}{(R-r)} - \frac{1}{R}\right] = k^2\frac{Sr\,(2R-r)}{R^2\,(R-r)^2}$$

Since r is very small compared to R, this is very approximately

$$(12) \qquad D = \frac{2k^2 Sr}{R^3}$$

It follows from Fig 42 and equation (5) that the disturbing acceleration in all parts of the orbit of the moon depends upon the distance of the sun from the earth in nearly the same way, therefore, the disturbing effect of the sun upon the moon varies nearly as the inverse third power of the distance of the sun from the earth

The average for a whole revolution of the earth is

$$(13) \qquad \bar{D} = \frac{4k^2 Sr}{P}\int_0^{\frac{P}{2}} \frac{dt}{R^3}$$

From the law of areas $dt = \dfrac{R^2 d\theta}{h}$, hence equation (13) becomes

$$(14) \quad \bar{D} = \frac{4k^2 Sr}{Ph}\int_0^{\pi}\frac{d\theta}{R} = \frac{4k^2 Sr}{Ph}\int_0^{\pi}\frac{(1+e\cos\theta)\,d\theta}{a\,(1-e^2)} = \frac{4k^2 Sr\,\pi}{Pha\,(1-e^2)}$$

But it is known that [Chap V, equations (22) and (27)]

$$h = k\sqrt{(1+m)\,a\,(1-e^2)},$$

$$P = \frac{2\pi a^{\frac{3}{2}}}{k\sqrt{1+m}},$$

therefore

$$(15) \qquad \bar{D} = \frac{2k^2 Sr}{a^3\,(1-e^2)^{\frac{3}{2}}}$$

As e decreases $\bar{D}$ decreases, therefore, as the eccentricity of the earth's orbit decreases, the efficiency of the sun in decreasing the attraction of the earth for the moon gradually decreases, and the mean motion of the moon increases correspondingly The changes are so small that the alteration in the orbit is almost inappreciable, but in the course of centuries the longitude of the moon is sensibly increased The theoretical amount of the acceleration is about 6″ in a century The amount derived from a discussion of eclipses varies from 8″ to 12″ It has been suggested that tidal retardation, lengthening the day, has caused the unexplained part of the apparent change, but the subject seems to be open yet to some question

The very long periodic variations in the eccentricity of the earth's orbit, whose effects upon the motion of the moon have just been considered, are due to the perturbations of the other planets Although their masses are so small and they are so remote that their direct perturbations of the moon's motion are almost insensible, yet they cause this and other important variations indirectly through their disturbances of the orbit of the earth This example of indirect action illustrates the great intricacy of the problem of the motions of the bodies of the solar system, and shows that methods of the greatest refinement must be employed in deriving satisfactory numerical results

156 The Variation There is another important perturbation in the motion of the moon which does not depend upon the eccentricity of its orbit It was discovered by Tycho Brahe, from observation, about 1590 Newton explained the cause of it in the *Principia* by a direct and elegant method which elicited the praise of Laplace

It can be explained most readily by supposing that the undisturbed motion of the moon is in a circle As has been shown, the normal component of the sun's disturbing acceleration is negative in the

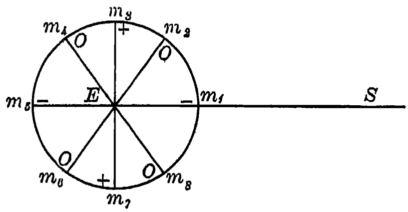

Fig 45

intervals $m_8m_1m_2$ and $m_4m_5m_6$, with maximum values at m_1 and m_5 Suppose the undisturbed motion at m_1 is in a circle, that is, that the acceleration due to the attraction of the earth exactly balances the centrifugal' acceleration There is no tangential component at this point but a large negative normal component The result is that the force which tends toward E is diminished and the orbit is less curved at this point than the circle Therefore the moon will recede to a greater distance from the earth in quadrature than in the circular orbit At the point m_3 the tangential component is zero, the force which tends toward E is increased, and the curvature is greater than in the circle The conditions vary continuously from those at m_1 to

those at m_3 in the interval m_1m_3. The corresponding changes in the remainder of the orbit are evident. The whole result is that the orbit is lengthened in the direction perpendicular to the line from the earth to the sun. If the sun be assumed to be so far distant that its disturbing effects in the interval $m_3m_5m_7$ are equal to those in the interval $m_7m_1m_3$, the orbit, under proper initial conditions, is symmetrical with respect to E as a center, and closely resembles an ellipse in form. This change of form of the orbit, and the auxiliary changes in the rate at which the radius vector sweeps over areas, give rise to an inequality in longitude between the mean position and the true position of the moon which amounts at times to about 35′, and is called the *variation*.

This perturbation has an interesting and important connection with the modern methods in the Lunar Theory, which were founded by G. W. Hill in his celebrated memoirs in the first volume of the *American Journal of Mathematics*, and in the *Acta Mathematica*, vol. VIII. A complete account of this method is given in Brown's *Lunar Theory* in the chapter entitled, *Method with Rectangular Coordinates*. Hill neglected the solar parallax; that is, he assumed that the disturbing force is equal in corresponding points in conjunction with, and opposition to, the sun. Instead of taking an ellipse as a first approximation, he took as an intermediate orbit that *variational orbit* which is closed with respect to axes rotating with the mean angular velocity of the sun, with a synodic period equal to the synodic period of the moon. The conception is not only one of great value, but the analysis was made by Hill with rare ingenuity and elegance.

157. The Parallactic Inequality. Since the sun is only a finite distance from the earth, its disturbing effects will not be exactly the same in points symmetrically situated with respect to the line m_3m_7, but will be greater on the side $m_7m_1m_3$. This introduces a distortion in the variational orbit, which leads to an inequality of about 2′ 7″ in the longitude of the moon compared to the theoretical position in the variational orbit. Since it is due to the parallax of the sun it has been called the *parallactic inequality*. Laplace remarked that, when it has been determined with very great accuracy from a long series of observations, it will furnish a satisfactory method of obtaining the distance of the sun. The chief practical difficulty is that the troublesome problem of finding the relative masses of the earth and moon must be solved before the method can be applied.*

* See Brown's *Lunar Theory*, p. 127.

158 The Motion of the Line of Apsides On account of the more complicated manner in which the different components affect the motion of the line of apsides, the perturbations of this element present greater difficulties than those heretofore considered Suppose first that the line of apsides coincides with the line ES, and that the perigee is at m_1 The normal component at m_1 is negative, and will, therefore (table, p 232), produce a retrogression of the line of apsides It was shown in Art 143 that the effectiveness of a normal component acting while the moon describes a short arc at apogee is to that of an equal normal component acting while an equal arc is described at perigee as $a(1+e)$ is to $a(1-e)$ Moreover, equation (12) shows that the total disturbing acceleration, and consequently the normal component, varies directly as the distance of the moon from the earth Therefore the normal component is greater at apogee, and is more effective in proportion to its magnitude than the corresponding acceleration at perigee The normal component is positive, though comparatively small, in the intervals $m_2m_3m_4$ and $m_6m_7m_8$ These intervals are almost equally divided by K and L (Fig 39), therefore it follows from the table that the total effect in these intervals is very small Hence the result in a whole revolution is to rotate the line of apsides forward through a large angle Similar reasoning leads to precisely the same results when the perigee is at m_5

The tangential component is equal in numerical value and opposite in sign on opposite sides of the major axis, and it follows from the table that it will not produce directly any permanent change in the position of the line of apsides for this configuration

Suppose now that the line of apsides is perpendicular to the line ES It is immaterial in this discussion at which end of the line the perigee is, but, to fix the ideas, it will be taken at m_3 The normal component is positive in the interval $m_2m_3m_4$, and, according to the table, rotates the line of apsides forward It is also positive in the interval $m_6m_7m_8$ and rotates the line of apsides backward In the latter case the disturbing acceleration is greater, and more effective for its magnitude, so that the whole result is a retrogression The intervals $m_8m_1m_2$ and $m_4m_5m_6$, in which the normal components are negative, are divided nearly equally by L and K, hence it is seen from the table that their results almost exactly balance each other in a whole revolution

The tangential component is positive in the interval m_3m_5 and negative in m_5m_7 From the table it is seen that a positive T in the

interval $m_3m_5m_7$ will cause the line of apsides to rotate forward, and a negative, backward Since the sign of T is opposite in the two nearly equal parts of the interval the whole result upon the line of apsides will be very small The result is the same in the half revolution

Tangential Component

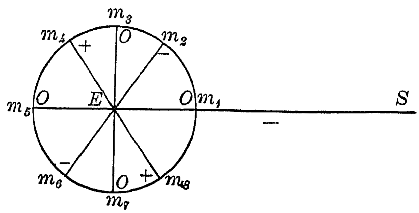

Fig 46

$m_7m_1m_3$ Thus it is seen that the combined effects of the normal and tangential components in a whole revolution is to rotate the line of apsides *backward* when it is orthogonal to the line from the earth to the sun

It was found that the line of apsides rotates *forward* when it coincides with the line from the earth to the sun The next question of importance is whether the progression or the retrogression is the greater It is noticed that the total changes arising from the action of the tangential components are the differences of nearly equal tendencies, and therefore small The same may be said of the normal components which act in the vicinity of the ends of the minor axis of the ellipse Moreover, in the two positions considered they act in opposite directions so that their whole result is still smaller The most important changes arise from the normal components which act in the vicinity of the ends of the major axis In the first case, in which the line of apsides progresses, they are about twice as great as in the second, in which the line of apsides regresses Therefore, the whole change for the two positions of the line of apsides is a progression The results for the positions near the two considered will be similar, but less in amount up to some intermediate points, where the rotation of the line of apsides in a whole revolution of the moon will be zero From the way in which the tangential components change sign (see Fig 46) it is evident that these points will be nearer to m_3 and m_7 than to m_1 and m_5, therefore *the average results for all possible positions is an advance in the line of apsides*

159 Secondary Effects The results thus far have been derived as though the sun were stationary It moves, however, in the same direction as the moon It has been shown that when the moon is near apogee and the sun near the line of apsides, the apsides advance This advance tends to preserve the relation of the orbit with reference to the position of the sun, and the advance of the apsides is prolonged and greatly increased On the other hand, when the moon is at perigee and the sun near the line of apsides the line of apsides moves backward, the sun moving one way and the line of apsides the other, this particular relation of the sun and the moon's orbit is quickly destroyed, and the retrogression is less than it would have been if the sun had remained stationary In a similar manner, for every relative position of the line of apsides, the progression is increased and the retrogression is decreased

There is another important secondary effect which depends upon the tangential component and is independent of the motion of the sun As an example, take the case in which the line of apsides passes through the sun with the perigee at m_1 The tangential component in m_3m_5 is positive, and, according to the table, rotates the line of apsides forward until the moon arrives at apogee But, as the line of apsides progresses, the moon will arrive at apogee later, and the effect of this component will be increased When the motion of the sun is also included this *secondary effect* becomes of still greater importance In this manner, perturbation exaggerates perturbation, and it is clear what astronomers mean when they say that nearly half the motion of the lunar perigee is due to the square of the disturbing force, or that it is obtained in a second approximation

The theoretical determination of the motion of the moon's line of apsides has been one of the most troublesome problems of Celestial Mechanics, the secondary effects escaped Newton when he wrote the *Principia* *, and were not explained to astronomers until Clairaut developed his Lunar Theory in 1749 The most successful and masterful analysis of the subject yet made is undoubtedly that of G W Hill, in the *Acta Mathematica*, vol VIII, which, for the terms treated, leaves nothing to be desired The line of apsides of the moon's orbit makes a complete revolution in about $9\frac{1}{2}$ years

* In the manuscripts which Newton left, and which are now known as the Portsmouth Collection, having been published but recently, a correct explanation of the motion of the line of apsides is given, and nearly correct numerical results are obtained

160 Perturbations of the Eccentricity Suppose the line of apsides passes through the sun and that the perigee is at m_1 From the symmetry of the normal components with respect to the line ES and the results given in the table, it follows that the increase and the decrease in the eccentricity in a complete revolution, due to this component, are exactly equal in this case From the way in which the tangential component changes sign, and from the results given in the table, it follows that the changes in the eccentricity, due to this component, also exactly balance Therefore there is no change in the eccentricity in a complete revolution of the moon under the conditions announced In a similar manner the same results are reached when the perigee is at m_5

Suppose the line of apsides has the direction m_3m_7 and that the perigee is at m_3 As before, neither the normal nor the tangential component makes any permanent change in the eccentricity

Consider the case at which the line of apsides has some intermediate position, as m_2m_6 Suppose the perigee is at m_2 Consider simultaneously with this the case in which the line of apsides is in a position symmetrically opposite to $m\ m_6$ with respect to the line m_3m_7, that is, the case in which the line of apsides is in m_4m_8 with the perigee at m_4 Consider first the effects of the normal component While the moon describes the arc m_2m_4 in the first, the eccentricity decreases, while it describes the arc $m\ m_4$ in the second, the eccentricity increases equally While the moon describes the arc m_4m_6 in the first, the eccentricity increases, while it describes the arc m_8m_2 in the second, it decreases equally In a similar manner every arc in the one may be paired with an arc in the second, so that the sum of the perturbations of the eccentricity while the moon is describing them is zero An analogous pairing is possible in the case of the tangential component Therefore in the two cases considered together the positive and negative changes in the eccentricity exactly neutralize each other All the possible positions of the major axis may be paired in this way

The sun does not, however, stand still while the moon makes its revolution, and the conditions announced can never be fulfilled Nevertheless, it is useful to show how the different configurations, even though changing from instant to instant, may be paired In a very great number of revolutions the supplementary configurations will have occurred an equal number of times, and the eccentricity will have returned to its original value The period required for this cycle of change depends in the first place upon the periods of the sun and the

moon, in the second place, upon the eccentricity of the sun's orbit (the earth's orbit), and lastly upon the manner in which the lines of apsides of the sun's and moon's orbits rotate *There is, therefore, no permanent change in the eccentricity of the moon's orbit**

The present system, with abundant Geological and Biological evidence of a very long existence for the earth in at least approximately its present condition, shows with reasonable certainty that the system is nearly stable, if not quite It is a remarkable fact, though, that those two elements, the line of nodes and the line of apsides, which may change continually in one direction without threatening the stability of the system do, on the average, respectively retrograde and advance forever

161 The Evection It has just been shown that the eccentricity does not change in the long run, yet it undergoes periodic variations of considerable magnitude which give rise to the largest lunar perturbation, known as the *evection* At its maximum effect it displaces the moon in geocentric longitude through an angle of about $1\frac{1}{4}$ compared to its position in the undisturbed elliptic orbit This variation was discovered by Hipparchus and was carefully observed by Ptolemy

The perturbations of the elements, and of the eccentricity in particular, depend upon two things, the position of the moon in its orbit, and the position of the moon with respect to the earth and sun Suppose the moon and sun start in conjunction with the perigee at m_1 Consider the motion throughout one synodic revolution It follows from the table and figures 43 and 44 that the eccentricity is not changing when the moon is at m_1, that it is decreasing when the moon is at m_2, m_3, and m_4, that it is not changing when the moon is at m_5, that it is increasing when the moon is at m_6, m_7, and m_8, and that it ceases to change when the moon has returned to m_1 again This is true only under the hypothesis that the perigee has remained at m_1 throughout the whole revolution, or, in other words, that the line of apsides advances as fast as the sun moves in its orbit Now, the actual case is that the sun moves about $8\frac{1}{2}$ times as fast as the line of apsides rotates Since the synodic period of the moon is about $29\frac{1}{2}$ days while the sun moves about one degree daily, the moon will be about 26° past its perigee when it arrives at m_1 What modification in the conclusions does this introduce? The normal component is positive

* The corresponding theorems are true regarding the orbits of the planets

and, in this part of the orbit, causes an increase in the eccentricity, while the tangential makes no change, since it is zero As the moon proceeds past m_1 the normal component becomes less in numerical value, while the tangential component becomes negative and tends to decrease the eccentricity The tendencies of the two components to change the eccentricity in opposite directions balance when the moon is at the same point between m_1 and m_2, after which the eccentricity decreases There is a corresponding lag in the point near m_5 at which the eccentricity ceases to decrease and begins to increase Similar conclusions are reached starting from any other initial configuration

The results may be summarized thus The perturbations of the sun decrease the eccentricity of the moon's orbit somewhat more than half of a synodical revolution, and then increase it for an equal time These changes in the eccentricity cause deviations in the geocentric longitude from the ones given by the elliptic theory, which constitute the *evection* The appropriate methods show that the period of this inequality is about $31\frac{4}{5}$ days

162 Gauss' Method of Computing Secular Variations It has been shown in the preceding articles that some of the elements, such as the line of nodes and the line of apsides, vary in one direction without limit This change is not at a uniform rate, for in addition to the general variations there are many short period oscillations which are of such magnitude that the element frequently varies in the opposite direction When the results are put into the symbols of analysis the general average advance is represented by a term proportional to the time, called the *secular variation*, while the deviations from this uniform change are represented by a sum of periodic terms having various periods and phases Thus it is seen that the secular variations are caused by a sort of average of the disturbing forces when the disturbing and disturbed bodies occupy every possible position with respect to each other

There are other elements, such as the inclination and the eccentricity which, though periodic in the long run, vary continuously in one direction on the average for many thousands of years These changes may be regarded as secular variations also, and they likewise result from a sort of average of perturbations

In 1818 Gauss published a memoir upon the theory of secular variations based upon the conceptions outlined above His method has been applied especially in the computation of the secular variations of the elements of the planetary orbits Instead of considering the motions

of the bodies, Gauss supposed that the mass of each planet is spread out in an elliptical ring coinciding with its orbit in such a manner that the density at each point is inversely as the velocity with which the body moves at that point He then showed how to compute the attraction of one ring upon the other, and the rate at which their positions and shapes would change These are the secular variations of the elements

The method has been the subject of quite a number of memoirs Probably the most useful for practical purposes is by G W Hill in vol I of the *Astronomical Papers of the American Ephemeris and Nautical Almanac* Hill's formulas have been applied by Professor Eric Doolittle with great success*, the results agreeing very closely with those found by Leverrier and Newcomb by entirely different methods

163 The Long Period Inequalities In the theories of the mutual perturbations of the planets very large terms of long period occur They arise only when the periods of the two bodies considered are nearly commensurable, and it is easy to discover their cause from geometrical considerations

Since the most important variation occurs in the mutual perturbations of Jupiter and Saturn the explanation will be adapted to that case Five times the period of Jupiter is a little more than twice the period of Saturn Suppose that the two planets are in conjunction at the

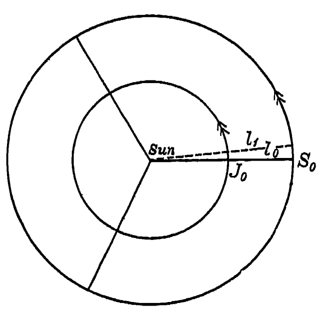

Fig 47

origin of time on the line l_0 After five revolutions of Jupiter and two of Saturn they will be in conjunction again on a line l_1 very near l_0, but having a little greater longitude This continues indefinitely, each

* *The Astronomical Journal*, 1896—1902

conjunction occurring at a little greater longitude than the preceding. Conjunctions occurring frequently at about the same points in the orbits cause very large perturbations, and the *Long Period* is the time which it takes the point of conjunction to make a complete revolution. In the case of Jupiter and Saturn it is about 918 years. This inequality, which is the greatest in the longitudes of the planets, displacing Jupiter 21′ and Saturn 49′, long baffled astronomers in their attempts to explain it as a necessary consequence of the law of gravitation. Laplace finally made one of his many important contributions to Celestial Mechanics by pointing out its true cause, and showing that theory and observation agree.

XXIII. PROBLEMS

1. Prove that the locus of the point at which the attractions of the sun and earth are equal is a sphere whose radius is $\frac{R\sqrt{SE}}{S-E}$, and whose center is on the line joining the sun and earth, at the distance $\frac{ER}{S-E}$ from the center of the earth opposite to the sun, where S and E represent the mass of the sun and earth respectively, and R the distance from the sun to the earth.

If $R=93,000,000$ miles, and $\frac{S}{E}=330,000$, then

$$\frac{R\sqrt{SE}}{S-E}=161550 \text{ miles},$$

$$\frac{RE}{S-E}=281 \text{ miles}.$$

Since the moon's orbit has a radius of about 240,000 miles, it is always attracted more by the sun than by the earth.

2. The moon may be regarded as revolving around the earth and disturbed by the sun, or as revolving around the sun and disturbed by the earth. Assume that the moon's orbit is a circle, and find the position at which the disturbing effects of the sun will be a maximum; show that the disturbing effects due to the earth, regarding the moon as revolving around the sun, are a minimum for the same position.

3. Find the ratio of the greatest disturbing effect of the sun to the least disturbing effect of the earth.

Ans. Let R equal the distance from the sun to the earth, ρ the distance from the sun to the moon, and r the distance from the earth to the moon; then

$$\frac{D_S}{D_E}=\frac{S}{E}\cdot\frac{r}{\rho^2}\cdot\frac{R^2-\rho^2}{R^2-r^2}=\frac{S}{E}\cdot\frac{r^3}{\rho^3}\cdot\frac{R+\rho}{R+r}=.0114$$

4 Find the ratio of the sun's disturbing force at its maximum value to the attraction of the sun, and to the attraction of the earth

$$Ans \quad \begin{cases} \dfrac{D_s}{A_s} = \dfrac{r(R+\rho)}{R^2} = 005, \\ \dfrac{D_s}{A_E} = \dfrac{S}{M}\dfrac{r^3(R+\rho)}{R^2\rho^2} = 011 \end{cases}$$

5 Suppose a planet disturbs the motion of another planet which is near to the sun Find the way in which all the elements of the orbit of the inner planet are changed for all relative positions of the bodies in their orbits

6 Show that, if the rates of change of the elements are known when the planet is in a particular position in its orbit, the intensity and direction of the disturbing force can be found Show that, if it is assumed that the distance of the disturbing body from the sun is known, its direction and mass can be found (This is part of the problem solved by Adams and Leverrier when they predicted the apparent position of Neptune from the knowledge of its perturbations of the motion of Uranus There are troublesome practical difficulties which arise on account of the minuteness of the quantities involved which do not appear in the simple statement given above)

HISTORICAL SKETCH AND BIBLIOGRAPHY

The first treatment of the Problem of Three Bodies, as well as of Two Bodies, was due to Newton It was given in the eleventh section of the *Principia*, and it was said by Airy to be "the most valuable chapter that was ever written on physical science" It contained a somewhat complete explanation of the Variation, the Parallactic Inequality, the Annual Equation, the Motion of the Perigee, the Variations in the Eccentricity, the Revolution of the Nodes, and the Variations in the Inclination The value of the motion of the lunar perigee found by Newton from theory was only half that given by observations In 1872, in certain of Newton's unpublished manuscripts, known as the Portsmouth Collection, it was found that Newton had accounted for the entire motion of the perigee by including perturbations of the second order (See Art 159) This work being unknown to astronomers, the motion of the lunar perigee was not derived from the theory of gravitation until 1749, when Clairaut found the true explanation, after being on the point of substituting for Newton's law of attraction one of the form $a=\frac{\mu}{r^2}+\frac{\nu}{r^3}$ Newton regarded the Lunar Theory as being very difficult, and he is said to have told his friend Halley in despair that it "made his head ache and kept him awake so often that he would think of it no more"

Since the days of Newton the methods of analysis have succeeded those of Geometry, except in elementary explanations of the causes of different sorts of perturbations In the eighteenth century the development of Lunar Theory, and of Celestial Mechanics in general, was almost entirely the work of five men Euler (1707—1783), a Swiss, born at Basle, living at St Petersburg from 1727 to 1747, at Berlin from 1747 to 1766, and at St Petersburg from 1766 to 1783, Clairaut (1713—1765), born at Paris, and spending nearly all his life in his native city, D'Alembert (1717—1783), also a native and an inhabitant of Paris, Lagrange (1736—1813), born at Turin, Italy, but of French descent, Professor of Mathematics in a military school in Turin from 1753 to 1766, succeeding Euler at Berlin and spending twenty years there, going to Paris and spending the remainder of his life in the French capital, and Laplace (1749—1827), son of a French peasant of Beaumont, in Normandy, Professor in the Ecole Militaire and in the Ecole Normale in Paris, where he spent most of his life after he was eighteen years of age The only part of their work which will be mentioned here will be that relating to the Lunar Theory The account of investigations in the general planetary theories comes more properly in the next chapter

There was a general demand for accurate lunar tables in the eighteenth century for the use of navigators in determining their positions at sea This, together with the fact that the motions of the moon presented the best test of the Newtonian Theory, induced the English Government and a number of scientific societies to offer very substantial prizes for lunar tables agreeing with observations within certain narrow limits Euler published some rather imperfect lunar tables in 1746 In 1747, Clairaut and D'Alembert presented to the Paris Academy on the same day memoirs on the Lunar Theory Each had trouble in explaining the motion of the perigee As has been stated, Clairaut found the source of the difficulty in 1749, and it was also discovered by both Euler and D'Alembert a little later Clairaut won the prize offered by the St Petersburg Academy in 1752 for his *Théorie de la Lune* Both he and D'Alembert published theories and numerical tables in 1754 They were revised and extended later Euler published a Lunar Theory in 1753, in the appendix of which the analytical method of the variation of the elements was partially worked out Tobias Mayer (1723—1762), of Gottingen, compared Euler's tables with observations and corrected them so successfully that he and Euler were each granted a reward of £3000 by the English Government In 1772 Euler published a second Lunar Theory which possessed many new features of great importance

Lagrange did little in the Lunar Theory except to point out general methods On the other hand, Laplace gave much attention to this subject, and made one of his important contributions to Celestial Mechanics in 1787, when he explained the cause of the secular acceleration of the moon's mean motion He also proposed to determine the distance of the sun from the parallactic inequality Laplace's theory is contained in the third volume of his *Mécanique Céleste*

Damoiseau (1768—1846) carried out Laplace's method to a high degree of approximation in 1824—28, and the tables which he constructed were used quite generally until Hansen's tables were constructed in 1857. Plana (1781—1869) published a theory in 1832, similar in most respects to that of Laplace. An incomplete theory was worked out by Lubbock (1803—1865) in 1830—4. A great advance along new lines was made by Hansen (1795—1874) in 1838 and again in 1862—4. His tables published in 1857 were very generally adopted for Nautical Almanacs. De Pontécoulant (1795—1874) published his *Théorie Analytique du Système du Monde* in 1846. The third volume contains his Lunar Theory worked out in detail. It is in its essentials similar to that of Lubbock. A new theory of great mathematical elegance, and carried out to a very high degree of approximation, was published by Delaunay (1816—1872) in 1860 and 1867.

A most remarkable new theory based on new conceptions, and developed by new mathematical methods, was published by G. W. Hill in 1878 in the *American Journal of Mathematics*. The first fundamental idea is to take the variational orbit as an approximate solution instead of the ellipse. A second approximation giving part of the motion of the perigee was published in volume VIII of *Acta Mathematica*. Hill's researches have been extended to higher approximations, and to a great extent completed, by a series of papers published by Professor E. W. Brown in the *American Journal of Mathematics*, vols. XIV, XV, and XVII, and in the *Monthly Notices of the R. A. S.*, LIII, LIV, and LV. The motion of the moon's nodes was found by Adams (1819—1892) by methods similar to those used by Hill in determining the motion of the perigee.

For the treatment of perturbations from geometrical considerations consult the *Principia*, Airy's (1801—1892) *Gravitation*, and Sir John Herschel's (1792—1871) *Outlines of Astronomy*. For the analytical treatment, aside from the original memoirs quoted, one cannot do better than to consult Tisserand's *Mécanique Céleste*, vol. III, and Brown's *Lunar Theory*. Both volumes are most excellent ones in both their contents and clearness of exposition. Brown's *Lunar Theory* especially is complete in those points, such as the meaning of the constants employed, which are apt to be somewhat obscure to one just entering this field.

CHAPTER IX

PERTURBATIONS—ANALYTICAL METHOD

164 The subject of the mutual perturbations of the motions of the heavenly bodies has been one to which many of the great mathematicians, from Newton's time on, have devoted a great deal of attention It is needless to say that the problem is of great difficulty and that many methods of attacking it have been devised Since the general solutions of the problem have not been obtained it has been necessary to treat special classes of perturbations by special methods It has been found convenient to divide the cases which arise in the solar system into three general classes, (*a*) the Lunar Theory and satellite theories, (*b*) the mutual perturbations of the planets, and (*c*) the perturbations of comets by planets The method which will be given in this chapter is applicable to the planetary theories, and it will be shown in the proper places why it is not applicable to the other cases References were given in the last chapter to treatises on the Lunar Theory, especially to Tisserand's and Brown's Some hints will be given in this chapter on the method of computing the perturbations of comets

The chief difficulties which arise in getting an understanding of the theories of perturbations come from the large number of variables which it is necessary to use, and the very long transformations which must be made in order to put the equations in a form suitable for actual computations It is not possible, because of the lack of space, to develop in detail the explicit expressions adapted to computation, and, indeed, it is not desired to emphasize this part, for it is much more important to get an accurate understanding of the nature of the problem, the mathematical features of the methods employed, the limitations which are necessary, the exact places where approximations are introduced, if at all, and their character, the origin of the various sorts of terms, and the foundations upon which the celebrated theorems

regarding the stability of the solar system rest Such is the object of this chapter

There are two general methods of considering perturbations, (a) as the variations of the coordinates of the various bodies, and (b) as the variations of the elements of their orbits These two conceptions were explained in the beginning of the last chapter Their analytical development was begun by Euler and Clairaut and was carried to a high degree of perfection by Lagrange and Laplace Yet there were points at which pure assumptions were made, it having become possible to establish completely the legitimacy of the proceedings, under the proper restrictions, only recently by the aid of the work in pure Mathematics of Cauchy, Weierstrass, and Poincaré *

It was seen that the fundamental concept in the method of the variation of parameters is that the body always moves in a conic section, but one in which the elements change from instant to instant The problem becomes the determination of the elements at any required time, that is, the elements are essentially the variables The first thing to be done is to transform the variables in the original differential equations by the equations which express them in terms of the new variables, the elements Then after the transformations are made the problem of integrating the equations arises again

165 Illustrative Example The complexity of the method in the Problem of Three Bodies, arising from the large number of variables involved and the intricate relations among them, is apt to obscure the theory and to make it unnecessarily difficult In order to avoid these difficulties many of the features of the method will be illustrated first by the simplest example possible

Suppose the differential equation to be solved is

$$(1) \qquad \frac{d^2x}{dt^2} + k^2x = mf\left(x, \frac{dx}{dt}, t\right)$$

If $m = 0$, this is the problem of the motion of a body attached to an elastic string (see Art 31) When m is not zero, the problem may be regarded the same with the addition of a perturbing force $mf\left(x, \frac{dx}{dt}, t\right)$ In order to make the treatment analogous to that of the Problem of Three Bodies let $\frac{dx}{dt} = x'$, then (1) is equivalent to

* F R Moulton in *Bulletin of the Am Math Soc*, June 1901

$$(2) \qquad \begin{cases} \dfrac{dx}{dt} - x' = 0, \\ \dfrac{dx}{dt} + k^2 x = mf(x, x', t) \end{cases}$$

The corresponding equations for undisturbed motion are

$$(2') \qquad \begin{cases} \dfrac{dx}{dt} - x' = 0, \\ \dfrac{dx'}{dt} + k^2 x = 0 \end{cases}$$

Equations (2′) are linear and may be solved by assuming the particular solutions

$$(3) \qquad \begin{cases} x = Ke^{\lambda t}, \\ x' = Le^{\lambda t} \end{cases}$$

Substituting in (2′) and dividing out the common factor $e^{\lambda t}$, there remain the equations

$$(4) \qquad \begin{cases} \lambda K - L = 0, \\ k^2 K + \lambda L = 0 \end{cases}$$

These homogeneous equations can be satisfied, except for $K = L = 0$, only if the determinant vanishes That is, if

$$(5) \qquad \begin{vmatrix} \lambda, & -1 \\ k, & \lambda \end{vmatrix} \equiv \lambda^2 + k^2 = 0$$

Therefore $\lambda_1 = \sqrt{-1}k$, $\lambda_2 = -\sqrt{-1}k$ in order that (3) may satisfy (2′) Since there are two values of λ there are two particular solutions of the type (3) Let the arbitrary constants be K_1 and K_2, and from (4) $L_1 = \lambda_1 K_1$, $L = \lambda_2 K_2$ Therefore the general solutions of (2′) are

$$(6) \qquad \begin{cases} x = K_1 e^{\sqrt{-1}kt} + K_2 e^{-\sqrt{-1}kt}, \\ x' = \lambda_1 K_1 e^{\sqrt{-1}kt} + \lambda \ K_2 e^{-\sqrt{-1}kt} \end{cases}$$

If the conditions at $t = 0$ are real, K_1 and K must have the form

$$\begin{cases} 2K_1 = a_1 - \sqrt{-1}a_2, \\ 2K_2 = a_1 + \sqrt{-1}a \end{cases}$$

Then equations (6) become in the trigonometrical form

$$(7) \qquad \begin{cases} x = a_1 \cos(kt) + a \ \sin(kt), \\ x' = -ka_1 \sin(kt) + ka \ \cos(kt) \end{cases}$$

Now consider equations (2) Let equations (7) be the equations of transformation expressing the old variables, x and x', in terms of the new variables, a_1 and a_2 From their form it is seen that they are

valid for all values of the time The transformation is accomplished by substituting them directly in (2) The first derivatives are

$$\begin{cases} \dfrac{dx}{dt} = -k\alpha_1 \sin(kt) + k\alpha_2 \cos(kt) + \cos(kt)\dfrac{d\alpha_1}{dt} + \sin(kt)\dfrac{d\alpha_2}{dt}, \\ \dfrac{dx'}{dt} = -k^2\alpha_1 \cos(kt) - k^2\alpha_2 \sin(kt) - k\sin(kt)\dfrac{d\alpha_1}{dt} + k\cos(kt)\dfrac{d\alpha_2}{dt} \end{cases}$$

Therefore

$$(8) \quad \begin{cases} \dfrac{dx}{dt} - x' \equiv \cos(kt)\dfrac{d\alpha_1}{dt} + \sin(kt)\dfrac{d\alpha_2}{dt} = 0, \\ \dfrac{dx'}{dt} + k^2 x \equiv -k\sin(kt)\dfrac{d\alpha_1}{dt} + k\cos(kt)\dfrac{d\alpha_2}{dt} = mf(x, x', t) \end{cases}$$

It is noticed that a large number of terms drop out because *the equations of transformation* (7) *are the solutions of the undisturbed problem* (2)

Solving (8), it is found that

$$(9) \quad \begin{cases} \dfrac{d\alpha_1}{dt} = -\dfrac{m}{k}\phi(\alpha_1, \alpha_2, t)\sin(kt), \\ \dfrac{d\alpha_2}{dt} = \dfrac{m}{k}\phi(\alpha_1, \alpha_2, t)\cos(kt), \end{cases}$$

where

$$\phi(\alpha_1, \alpha_2, t) = f(x, x', t)$$

If ϕ did not contain α_1 and α_2 the problem would be reduced to quadratures and the integration could be performed In order to simplify the treatment from this point on, suppose α_1 and α_2 do not occur in ϕ, and, in order to illustrate terms of different kinds, suppose

$$\phi = \sin(kt)$$

Then the integrals of (9) are found to be

$$(10) \quad \begin{cases} \alpha_1 = -\dfrac{m}{2k}t + \dfrac{m}{4k^2}\sin(2kt) + \alpha_1', \\ \alpha_2 = -\dfrac{m}{4k^2}\cos(2kt) + \alpha_2', \end{cases}$$

where α_1' and α_2' are the constants of integration When these expressions for α_1 and α_2 are substituted in (7) the latter are the general solutions of (1), after $f\left(x, \dfrac{dx}{dt}, t\right)$ has been replaced by $\sin(kt)$

The element α_1 depends upon two terms, if m is positive the first decreases indefinitely with the time and gives rise to what is called a

secular variation in α_1, the second term is periodic and gives rise to what is called a *periodic variation* If the secular term alone is considered the graphical representation of the element is the straight

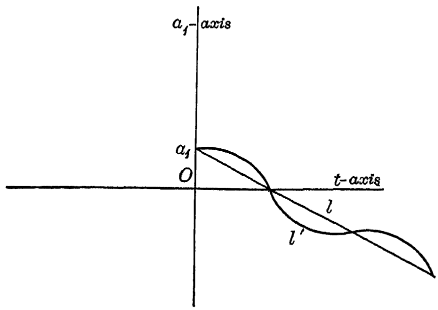

Fig 48

line l If the periodic term is added it becomes the wavy line l' The element a has no secular term

166 Equations in the Problem of Three Bodies Consider the motion of two bodies, m_1 and m_2, with respect to the sun, S Take the center of the sun as origin and let the coordinates of m_1 be x_1, y_1, z_1, and of m_2, x_2, y_2, z_2 Let the distances of m_1 and m_2 from the sun be r_1 and r_2 respectively, and the distance from m_1 to m_2, r_{12} Then the differential equations of motion, as derived in Art 112, are

$$
(11)\quad \begin{cases}
\dfrac{d^2x_1}{dt^2} + k^2(S+m_1)\dfrac{x_1}{r_1^3} = m_2\dfrac{\partial R_{12}}{\partial x_1}, \\[2mm]
\dfrac{d^2y_1}{dt^2} + k^2(S+m_1)\dfrac{y_1}{r_1^3} = m_2\dfrac{\partial R_{12}}{\partial y_1}, \\[2mm]
\dfrac{d^2z_1}{dt^2} + k^2(S+m_1)\dfrac{z_1}{r_1^3} = m_2\dfrac{\partial R_{12}}{\partial z_1}, \\[2mm]
\dfrac{d^2x_2}{dt^2} + k^2(S+m_2)\dfrac{x_2}{r_2^3} = m_1\dfrac{\partial R_{21}}{\partial x_2}, \\[2mm]
\dfrac{d^2y_2}{dt^2} + k^2(S+m_2)\dfrac{y_2}{r_2^3} = m_1\dfrac{\partial R_{21}}{\partial y_2}, \\[2mm]
\dfrac{d^2z_2}{dt^2} + k^2(S+m_2)\dfrac{z_2}{r_2^3} = m_1\dfrac{\partial R_{21}}{\partial z_2}, \\[2mm]
R_{12} = k^2\left[\dfrac{1}{r_{12}} - \dfrac{x_1x_2+y_1y_2+z_1z_2}{r_2^3}\right], \\[2mm]
R_{21} = k^2\left[\dfrac{1}{r_{12}} - \dfrac{x_2x_1+y_2y_1+z_2z_1}{r_1^3}\right]
\end{cases}
$$

The right members of these differential equations are multiplied by the very small factors m_1 and m_2, therefore they will be of slight importance, at least for a considerable time If m_1 and m_2 are put equal to zero in the right members, the first three equations and the second three become independent of each other, and the problem for each set of three equations reduces to that of two bodies, and can be completely solved

It will be advantageous to reduce the six equations (11) of the second order to twelve of the first order Let

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt}, \quad z' = \frac{dz}{dt},$$

then equations (11) become

$$(12) \qquad \begin{cases} \dfrac{dx_1}{dt} - x_1' = 0, \quad \dfrac{dx_1'}{dt} + k^2 (S + m_1) \dfrac{x_1}{r_1^3} = m_2 \dfrac{\partial R_{1\,2}}{\partial x_1}, \\[2ex] \dfrac{dy_1}{dt} - y_1' = 0, \quad \dfrac{dy_1'}{dt} + k^2 (S + m_1) \dfrac{y_1}{r_1^3} = m_2 \dfrac{\partial R_{1\,2}}{\partial y_1}, \\[2ex] \dfrac{dz_1}{dt} - z_1' = 0, \quad \dfrac{dz_1'}{dt} + k^2 (S + m_1) \dfrac{z_1}{r_1^3} = m_2 \dfrac{\partial R_{1\,2}}{\partial z_1}, \end{cases}$$

and similar equations in which the subscript is 2

When the motions of m_1 and m_2 are undisturbed by each other these equations become

$$(13) \qquad \begin{cases} \dfrac{dx_1}{dt} - x_1' = 0, \quad \dfrac{dx_1'}{dt} + k^2 (S + m_1) \dfrac{x_1}{r_1^3} = 0, \\[2ex] \dfrac{dy_1}{dt} - y_1' = 0, \quad \dfrac{dy_1'}{dt} + k^2 (S + m_1) \dfrac{y_1}{r_1^3} = 0, \\[2ex] \dfrac{dz_1}{dt} - z_1' = 0, \quad \dfrac{dz_1'}{dt} + k^2 (S + m_1) \dfrac{z_1}{r_1^3} = 0, \end{cases}$$

and an independent system of similar equations in which the subscript is 2 Let $\Omega_1 = \frac{1}{2}(x_1'^2 + y_1'^2 + z_1'^2) - k^2 \dfrac{(S + m_1)}{r_1}$, then equations (13) may be written

$$(14) \qquad \begin{cases} \dfrac{dx_1}{dt} = \dfrac{\partial \Omega_1}{\partial x_1'}, \quad \dfrac{dx_1'}{dt} = -\dfrac{\partial \Omega_1}{\partial x_1}, \\[2ex] \dfrac{dy_1}{dt} = \dfrac{\partial \Omega_1}{\partial y_1'}, \quad \dfrac{dy_1'}{dt} = -\dfrac{\partial \Omega_1}{\partial y_1}, \\[2ex] \dfrac{dz_1}{dt} = \dfrac{\partial \Omega_1}{\partial z_1'}, \quad \dfrac{dz_1'}{dt} = -\dfrac{\partial \Omega_1}{\partial z_1} \end{cases}$$

It is impossible not to notice one important difference between this problem and the simple one which was solved to illustrate the method The right member in the illustrative example was supposed to be a function of t alone before the integration was made, and there was no difficulty in obtaining the general and exact solution of the differential equation Similarly, if the right members of (12) were functions of t alone there would be no trouble in obtaining rigorous solutions, but they depend upon the coordinates of the moving bodies, and will, therefore, after the transformation, contain the new variables in a very complicated manner They enter in such a fashion that it is as difficult to obtain rigorous solutions after the transformations of variables as before

167 Transformation of Variables In order to avoid confusion in the analysis, and to be able to say where and how the approximations are introduced, the method of the variation of parameters must be regarded in the first instance as simply a transformation of variables, which is perfectly legitimate for all values of the time for which the equations of transformation are valid From this point of view the whole process is mathematically simple and lucid, the only trouble arising from the number of variables involved and the complicated relations among them

In Chapter V it was shown how to express the coordinates in the Problem of Two Bodies in terms of the elements and the time Let $\alpha_1, \cdots, \alpha_6$ represent the elements of the orbit m_1, and $\beta_1, \cdots, \beta_6$ those of m_2 Then the equations for the coordinates in the Problem of Two Bodies may be written

$$(15)\qquad \begin{cases} x_1 = f(\alpha_1, \cdots, \alpha_6, t), & x_1' = \theta(\alpha_1, \cdots, \alpha_6, t), \\ y_1 = g(\alpha_1, \cdots, \alpha_6, t), & y_1' = \phi(\alpha_1, \cdots, \alpha_6, t), \\ z_1 = h(\alpha_1, \cdots, \alpha_6, t), & z_1' = \psi(\alpha_1, \cdots, \alpha_6, t), \\ x_2 = f(\beta_1, \cdots, \beta_6, t), & x_2' = \theta(\beta_1, \cdots, \beta_6, t), \\ y_2 = g(\beta_1, \cdots, \beta_6, t), & y_2' = \phi(\beta_1, \cdots, \beta_6, t), \\ z_2 = h(\beta_1, \cdots, \beta_6, t), & z_2' = \psi(\beta_1, \cdots, \beta_6, t) \end{cases}$$

A transformation of variables in equations (12) will now be made Let it be forgotten for the moment that equations (15) are the solutions of the Problem of Two Bodies, and that the α_i and β_i are the elements of the two orbits, but let them be the equations which transform equations (12) in the old variables, x_1, y_1, z_1, x_1', y_1', z_1', x_2, y_2, z_2, x_2', y_2', z_2', into an equivalent system in the new variables,

$\alpha_1, \cdots, \alpha_6, \beta_1, \cdots, \beta_6$ It will be advantageous to make such transformations that they will be valid for all values of t, and it may be supposed that this condition is fulfilled The transformations are effected by computing the derivatives occurring in (12) and making direct substitutions The derivatives of equations (15) with respect to t are

$$
(16) \qquad \left\{
\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial x_1}{\partial t} + \sum_{i=1}^{6} \frac{\partial x_1}{\partial \alpha_i} \frac{d\alpha_i}{dt}, \\
\frac{dx_1'}{dt} &= \frac{\partial x_1'}{\partial t} + \sum_{i=1}^{6} \frac{\partial x_1}{\partial \alpha_i} \frac{d\alpha_i}{dt},
\end{aligned}
\right.
$$

The direct substitution of (16) in (12) gives

$$
(17) \qquad \left\{
\begin{aligned}
&\frac{\partial x_1}{\partial t} - x_1' + \sum_{i=1}^{6} \frac{\partial x_1}{\partial \alpha_i} \frac{d\alpha_i}{dt} = 0, \\
&\frac{\partial y_1}{\partial t} - y_1' + \sum_{i=1}^{6} \frac{\partial y_1}{\partial \alpha_i} \frac{d\alpha_i}{dt} = 0, \\
&\frac{\partial z_1}{\partial t} - z_1' + \sum_{i=1}^{6} \frac{\partial z_1}{\partial \alpha_i} \frac{d\alpha_i}{dt} = 0, \\
&\frac{\partial x_1'}{\partial t} + k^2 (S + m_1) \frac{x_1}{r_1^3} + \sum_{i=1}^{6} \frac{\partial x_1'}{\partial \alpha_i} \frac{d\alpha_i}{dt} = m_2 \frac{\partial R_1}{\partial x_1}, \\
&\frac{\partial y_1'}{\partial t} + k^2 (S + m_1) \frac{y_1}{r_1^3} + \sum_{i=1}^{6} \frac{\partial y_1'}{\partial \alpha_i} \frac{d\alpha_i}{dt} = m_2 \frac{\partial R_1}{\partial y_1}, \\
&\frac{\partial z_1'}{\partial t} + k^2 (S + m_1) \frac{z_1}{r_1^3} + \sum_{i=1}^{6} \frac{\partial z_1'}{\partial \alpha_i} \frac{d\alpha_i}{dt} = m_2 \frac{\partial R_1}{\partial z_1},
\end{aligned}
\right.
$$

and similar equations in x_2, y_2, z_2, and $\beta_1, \cdots, \beta_6$ These equations are linear in $\frac{d\alpha_i}{dt}$ and may be solved for these derivatives, expressing them in terms of $\alpha_1, \cdots, \alpha_6$, $\beta_1, \cdots, \beta_6$, and t

But if equations (15) are the solution of the problem of undisturbed elliptic motion in which the osculating orbits do not intersect, all the conditions so far imposed will be fulfilled, and the equations (17) will be greatly simplified Suppose (15) are the solutions of the equations for undisturbed motion It is seen from (13) that when $\alpha_1, \cdots, \alpha_6$ are constant $\frac{dx_1}{dt} - x_1' \equiv 0$ for all values of $\alpha_1, \cdots, \alpha_6$ such that the two bodies do not collide, and for all values of t The partial derivative $\frac{\partial x_1}{\partial t}$, when $\alpha_1, \cdots, \alpha_6$ are regarded as variables, is identical with $\frac{dx_1}{dt}$ when they are not Therefore $\frac{\partial x_1}{\partial t} - x_1' \equiv 0$, and similarly $\frac{\partial x_1'}{\partial t} + k^2 (S + m_1) \frac{x_1}{r_1^3} \equiv 0$, and

similar equations in y and z As a consequence of these relations equations (17) reduce to

$$(18)\quad \begin{cases} \sum_{i=1}^{6} \frac{\partial x_1}{\partial \alpha_i}\frac{d\alpha_i}{dt} = 0, & \sum_{i=1}^{6} \frac{\partial x_1'}{\partial \alpha_i}\frac{d\alpha_i}{dt} = m_2 \frac{\partial R_{1\,2}}{\partial x_1}, \\ \sum_{i=1}^{6} \frac{\partial y_1}{\partial \alpha_i}\frac{d\alpha_i}{dt} = 0, & \sum_{i=1}^{6} \frac{\partial y_1'}{\partial \alpha_i}\frac{d\alpha_i}{dt} = m_2 \frac{\partial R_{1\,2}}{\partial y_1}, \\ \sum_{i=1}^{6} \frac{\partial z_1}{\partial \alpha_i}\frac{d\alpha_i}{dt} = 0, & \sum_{i=1}^{6} \frac{\partial z_1'}{\partial \alpha_i}\frac{d\alpha_i}{dt} = m \frac{\partial R_{1\,2}}{\partial z_1}, \end{cases}$$

and similar equations for m_2 These equations are linear in $\frac{d\alpha_i}{dt}$ and may be solved for these derivatives unless the determinant is zero But the determinant of the linear system (18) is the Jacobian of the first set of equations (15) with respect to $\alpha_1, \quad , \alpha_6$, and cannot vanish if these functions are independent and give a simple and unique determination of the elements* But they are independent and in general give simple and unique values for the elements since they are the expressions for the coordinates in the Problem of Two Bodies

If $m_2 = 0$ the equations are linear and homogeneous, and since the determinant does not vanish they can be satisfied only by $\frac{d\alpha_i}{dt} = 0$, $(i = 1, \quad , 6)$ That is, the elements are constants, which, of course, is nothing new

Solving (18), it is found that

$$(19)\quad \begin{cases} \frac{d\alpha_i}{dt} = m\ \phi_i(\alpha_1, \quad , \alpha_6, \beta_1, \quad , \beta_6, t), & (i = 1, \quad , 6), \\ \frac{d\beta_i}{dt} = m_1 \psi_i(\alpha_1, \quad , \alpha_6, \beta_1, \quad , \beta_6, t), & (i = 1, \quad , 6) \end{cases}$$

It will be remembered that in determining the coordinates in the Problem of Two Bodies the first step viz the computation of the mean anomaly, involved the mean motion, defined by the equation

$$n_i = \frac{k\sqrt{S + m_i}}{a_i^{\frac{3}{2}}}$$

Since the n_i involve the masses of the planets the right members of (15), and consequently of (19), involve m_1 and m_2 implicitly

At this point a question of some delicacy arises and must be considered before the methods which follow can be justified The sums $S + m_1$ and $S + m_2$ in the left members of equations (11) will be

* See Baltzer's *Determinanten*, p 141

regarded as being fixed numbers independent of the attraction which m_1 and m_2 exert upon each other Of course, they are not independent of the mutual attraction of these bodies in the actual case, but they will be considered as being so in order to preserve the logic of the processes employed in the next articles It is clear that there is no logical difficulty in the matter, for a problem can at once be set up which will demand equations of the same form and possessing these properties It is only necessary to suppose that the relative accelerations of the sun upon m_1 and m_2 are $\frac{k^2(S+m_1)}{r_1^2}$ and $\frac{k^2(S+m_2)}{r_2^2}$, and that the planets are composed of such material that their mutual attraction is $\frac{k^2 m_1' m_2'}{r^2}$, where m_1' and m_2' are any numbers independent of m_1 and m_2 Such problems would arise if gravitation were selective The perturbations can be computed for all values of m_1' and m_2' leaving m_1 and m_2 fixed, then, if m_1' and m_2' are put equal to m_1 and m_2 respectively, the results reduce to those of the actual case under consideration

Hence, the values of the masses m_1 and m_2 entering implicitly in equations (15) and (19) are treated as fixed numbers, given in advance, and do not need to be retained explicitly, on the other hand, the m_1 and m_2 which are factors of the perturbing terms of the equations are retained explicitly, being supposed capable of taking any values not exceeding certain limits

168 Method of Solution Equations (11) are the general differential equations of motion for the Problem of Three Bodies Equations (12) are equally general No approximations were introduced in making the transformation of variables by (15), therefore equations (19) are general and rigorous The difference is that if (19) were integrated the elements would be found instead of the coordinates as in (11), but as the latter can always be found from the former this must be regarded as the solution of the problem

Instead of interrupting the course of mathematical reasoning by working out the explicit forms of (19), it will be preferable to show first by what methods they are solved Explicit mention will be made at the appropriate times of all points at which assumptions or approximations are made

When m_1 and m_2 are very small compared to S, as they are in the solar system, the orbits are very nearly fixed ellipses, and therefore α_i and β_i change very slowly It is *assumed* here that α_i and β_i may

be expressed in convergent power series in m_1 and m_2, as

$$(20)\qquad \begin{cases} \alpha_i = \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \alpha_i^{(j\, k)} m_1^j m_2^k, \\ \beta_i = \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \beta_i^{(j\, k)} m_1^j m_2^k, \end{cases}$$

where the upper indices of the α_i and β_i simply indicate the order of the coefficient The $\alpha_i^{(j\, k)}$ and $\beta_i^{(j\, k)}$ are functions of the time which are to be determined

It has been customary in the theory of perturbations to assume without proof that this expansion is valid for any desired length of time It can be proved that it is valid for a sufficiently small interval of time*, but as the method of demonstration gives only a limit within which the series certainly converge, and not the longest time during which they converge, and as the limit is almost certainly far too small it has never been computed It is to be understood, therefore, that the method which is just to be explained, is valid for a certain interval of time, which in the planetary theories is doubtless several hundreds of years Beyond this limit the values of the elements are not given by equations (20) because, from a mathematical point of view, the series diverge, yet, for practical purposes they may be useful The first terms of the series decrease with great rapidity, and if the stop in the summation is made soon enough the numerical values of the functions may be obtained with great accuracy†

Substituting (20) in (19) and developing with respect to m_1 and m_2, it is found that

$$(21)\qquad \begin{cases} \dfrac{d\alpha_i^{(0\,0)}}{dt} + \dfrac{d\alpha_i^{(0\,1)}}{dt} m_2 + \dfrac{d\alpha_i^{(1\,0)}}{dt} m_1 + \dfrac{d\alpha_i^{(1\,1)}}{dt} m_1 m_2 + \dfrac{d\alpha_i^{(0\,2)}}{dt} m_2^2 + \dfrac{d\alpha_i^{(2\,0)}}{dt} m_1^2 + \\ \quad = m_2 \phi_i(\alpha_1^{(0\,0)}, \ldots, \alpha_6^{(0\,0)}, \beta_1^{(0\,0)}, \ldots, \beta_6^{(0\,0)}, t) + m_2 \sum\limits_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \alpha_j} (\alpha_j^{(0\,1)} m_2 + \alpha_j^{(1\,0)} m_1) \\ \qquad + m_2 \sum\limits_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \beta_j} (\beta_j^{(0\,1)} m_2 + \beta_j^{(1\,0)} m_1) + \text{higher powers in } m_1 \text{ and } m_2, \\ \dfrac{d\beta_i^{(0\,0)}}{dt} + \dfrac{d\beta_i^{(0\,1)}}{dt} m_2 + \dfrac{d\beta_i^{(1\,0)}}{dt} m_1 + \dfrac{d\beta_i^{(1\,1)}}{dt} m_1 m_2 + \dfrac{d\beta_i^{(0\,2)}}{dt} m_2^2 + \dfrac{d\beta_i^{(2\,0)}}{dt} m_1^2 + \\ \quad = m_1 \psi_i(\alpha_1^{(0\,0)}, \ldots, \alpha_6^{(0\,0)}, \beta_1^{(0\,0)}, \ldots, \beta_6^{(0\,0)}, t) + m_1 \sum\limits_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \alpha_j} (\alpha_j^{(0\,1)} m_2 + \alpha_j^{(1\,0)} m_1) \\ \qquad + m_1 \sum\limits_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \beta_j} (\beta_j^{(0\,1)} m_2 + \beta_j^{(1\,0)} m_1) + \text{higher powers in } m_1 \text{ and } m_2, \\ \qquad (i = 1, \ldots, 6) \end{cases}$$

* See paper by F R Moulton, *Bulletin of Am Math Soc* June 1901

† See Poincaré s *Methodes Nouvelles* chap VIII for treatment of an analogous question

In the partial derivatives it is to be understood that α_i and β_i are replaced by $\alpha_i^{(0\ 0)}$ and $\beta_i^{(0\ 0)}$ respectively If m_1 and m_2 were not regarded as fixed numbers in the left members of equations (11) ϕ_i, ψ_i, $\frac{\partial \phi_i}{\partial \alpha_j}$, $\frac{\partial \phi_i}{\partial \beta_j}$, etc would have to be developed as power series in m_1 and m_2, thus adding greatly to the complexity of the work.

Within the limits of convergence the coefficients of like powers of m_1 and m_2 on the two sides of the equations are equal Hence, equating them, it follows that

$$(22)\qquad \begin{cases} \dfrac{d\alpha_i^{(0\ 0)}}{dt} = 0, \qquad (i=1, \ \ , 6), \\ \dfrac{d\beta_i^{(0\ 0)}}{dt} = 0 \end{cases}$$

$$(23)\qquad \begin{cases} \dfrac{d\alpha_i^{(0,\ 1)}}{dt} = \phi_i\,(\alpha_1^{(0\ 0)}, \ \ , \alpha_6^{(0\ 0)}, \beta_1^{(0\ 0)}, \ \ , \beta_6^{(0\ 0)}, t), \\ \dfrac{d\alpha_i^{(1\ 0)}}{dt} = 0, \\ \dfrac{d\beta_i^{(0\ 1)}}{dt} = 0, \\ \dfrac{d\beta_i^{(1\ 0)}}{dt} = \psi_i\,(\alpha_1^{(0\ 0)}, \ \ , \alpha_6^{(0,\ 0)}, \beta_1^{(0\ 0)}, \ \ , \beta_6^{(0\ 0)}, t) \end{cases}$$

$$(24)\qquad \begin{cases} \dfrac{d\alpha_i^{(1\ 1)}}{dt} = \sum_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \alpha_j}\, \alpha_j^{(1\ 0)} + \sum_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \beta_j}\, \beta_j^{(1\ 0)}, \\ \dfrac{d\alpha_i^{(0\ 2)}}{dt} = \sum_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \alpha_j}\, \alpha_j^{(0\ 1)} + \sum_{j=1}^{6} \dfrac{\partial \phi_i}{\partial \beta_j}\, \beta_j^{(0\ 1)}, \\ \dfrac{d\alpha_i^{(2,\ 0)}}{dt} = 0, \\ \dfrac{d\beta_i^{(1\ 1)}}{dt} = \sum_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \alpha_j}\, \alpha_j^{(0\ 1)} + \sum_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \beta_j}\, \beta_j^{(0\ 1)}, \\ \dfrac{d\beta_i^{(2,\ 0)}}{dt} = \sum_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \alpha_j}\, \alpha_j^{(1\ 0)} + \sum_{j=1}^{6} \dfrac{\partial \psi_i}{\partial \beta_j}\, \beta_j^{(1\ 0)}, \\ \dfrac{d\beta_i^{(0\ 2)}}{dt} = 0, \end{cases}$$

etc

Integrating (22) and substituting the values of $\alpha_i^{(0\ 0)}$ and $\beta_i^{(0\ 0)}$ thus obtained in (23) the latter are reduced to quadratures, and can be integrated, integrating (23) and substituting the expressions for $\alpha_i^{(0\ 1)}$,

$\alpha_i^{(1\,0)}$, $\beta_i^{(0\,1)}$, $\beta_i^{(1\,0)}$ in (24) the latter are reduced to quadratures, and can be integrated, and this process may be continued indefinitely In this manner the coefficients of the series (20) may be determined, and the values of α_i and β_i found to any desired degree of precision for values of the time for which the series converge

169 Determination of the Constants of Integration A new constant of integration is introduced when equations (22), (23), are integrated for each $\alpha_i^{(j\,k)}$, $\beta_i^{(j\,k)}$ These constants must now be determined

Let the constant which is introduced with the term $\alpha_i^{(j\,k)}$ be $a_i^{(j\,k)}$ and with $\beta_i^{(j\,k)}$, $b_i^{(j\,k)}$ It is seen from the differential equations that

$$\alpha_i^{(j\,0)} = a_i^{(j\,0)}, \qquad (j=0, \quad , \infty),$$
$$\beta_i^{(0\,k)} = b_i^{(0\,k)}, \qquad (k=0, \quad , \infty)$$

Instead of writing $\alpha_i^{(j\,k)}$ and $\beta_i^{(j\,k)}$ for the coefficients in (20), their constant and variable parts may be separated, after which they will have the form $\alpha_i^{(j\,k)} - a_i^{(j\,k)}$ and $\beta_i^{(j\,k)} - b_i^{(j\,k)}$ Then equations (20) may be written

$$(25)\qquad \begin{cases} \alpha_i = \sum\limits_{j=0}^{\infty} a_i^{(j\,0)} m_1^j + \sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{\infty} (\alpha_i^{(j\,k)} - a_i^{(j\,k)})\, m_1^j m_2^k, \\ \beta_i = \sum\limits_{k=0}^{\infty} b_i^{(0\,k)} m_2^k + \sum\limits_{j=1}^{\infty}\sum\limits_{k=0}^{\infty} (\beta_i^{(j\,k)} - b_i^{(j\,k)})\, m_1^j m_2^k \end{cases}$$

Let the values of α_i and β_i at $t=t_0$ be $\alpha_i^{(0)}$ and $\beta_i^{(0)}$ respectively Then at $t=t_0$,

$$\begin{cases} \alpha_i^{(0)} = \sum\limits_{j=0}^{\infty} a_i^{(j\,0)} m_1^j + \sum\limits_{j=0}^{\infty}\sum\limits_{k=1}^{\infty} (\alpha_i^{(j\,k)} - a_i^{(j\,k)})_{t=t_0} m_1^j m_2^k, \\ \beta_i^{(0)} = \sum\limits_{k=0}^{\infty} b_i^{(0\,k)} m_2^k + \sum\limits_{j=1}^{\infty}\sum\limits_{k=0}^{\infty} (\beta_i^{(j\,k)} - b_i^{(j\,k)})_{t=t_0} m_1^j m_2^k \end{cases}$$

Since these equations must be true for all values of m_1 and m_2 below certain limits, the coefficients of corresponding powers of m_1 and m_2 in the right and left members are equal, whence

$$(26)\qquad \begin{cases} a_i^{(0\,0)} = \alpha_i^{(0)}, \quad a_i^{(j\,0)} = 0, & (j=1, \quad , \infty), \\ b_i^{(0\,0)} = \beta_i^{(0)}, \quad b_i^{(0\,k)} = 0, & (k=1, \quad , \infty), \\ (\alpha_i^{(j\,k)})_{t=t_0} - a_i^{(j\,k)} = 0, & (j=1, \quad , \infty, \quad k=1, \quad , \infty), \\ (\beta_i^{(j\,k)})_{t=t_0} - b_i^{(j\,k)} = 0, & (j=1, \quad , \infty, \quad k=1, \quad , \infty) \end{cases}$$

Since all the terms of the right members except the first vanish at $t=t_0$, it follows that $a_i^{(0\,0)}$ and $\beta_i^{(0\,0)}$ are the osculating elements of the orbits of m_1 and m_2 respectively at the time $t=t_0$, and that the other coefficients of (20) are the definite integrals of the differential equations which define them taken between the limits $t=t_0$ and $t=t$

170 The Terms of the First Order The terms of the first order with respect to the masses are given by (23) Since the terms of order zero are the osculating elements at t_0, the equations become

$$(27)\qquad \left\{\begin{aligned} \frac{d\alpha_i^{(0,1)}}{dt} &= \phi_i\,(\alpha_1^{(0)}, \quad, \alpha_6^{(0)}, \beta_1^{(0)}, \quad, \beta_6^{(0)}, t), \\ \frac{d\beta_i^{(1\,0)}}{dt} &= \psi_i\,(\alpha_1^{(0)}, \quad, \alpha_6^{(0)}, \beta_1^{(0)}, \quad, \beta_6^{(0)}, t) \end{aligned}\right.$$

The right members of these equations are proportional to the rates at which the various elements of the orbits of the two planets would vary at any time t, if the two planets were moving at that instant strictly in the original ellipses The integrals of (27) are, therefore, the summations of the instantaneous effects, or, in other words, they are the summations of the changes which would be produced if the forces and their instantaneous results were always exactly equal to those in the undisturbed orbits Of course the perturbations modify these conditions and produce secondary, tertiary, and higher order effects They are included in the coefficients of higher powers of m_1, m_2 in (20)

The quantities $\alpha_i^{(0\,1)}$ and $\beta_i^{(1\,0)}$ are usually called perturbations of the first order with respect to the masses The reason is clearly because they are the coefficients of the first powers of the masses in the series (20) In the planetary theories it is not necessary to go to perturbations of higher orders except in the case of the larger planets which are near each other, and then comparatively few terms are large enough to be sensible It is not necessary in the present state of the planetary theories to include terms of the third order except in the mutual perturbations of Jupiter and Saturn

Instead of there being but two planets and the sun there are eight planets and the sun, so that the actual theory is not quite so simple as that outlined above Yet, as will be shown, the increased complexity comes chiefly in the perturbations of higher orders If there were a third planet m_3 with elements $\gamma_1, \quad, \gamma_6$ equations (23) would become

$$
(28) \quad \left\{
\begin{aligned}
\frac{d\alpha_i^{(1\,0\,0)}}{dt} &= 0, \\
\frac{d\alpha_i^{(0\,1\,0)}}{dt} &= \phi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t), \\
\frac{d\alpha_i^{(0\,0\,1)}}{dt} &= \phi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \gamma_1^{(0)}, \quad , \gamma_6^{(0)}, t), \\
\frac{d\beta_i^{(1\,0\,0)}}{dt} &= \psi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t), \\
\frac{d\beta_i^{(0\,1\,0)}}{dt} &= 0, \\
\frac{d\beta_i^{(0\,0\,1)}}{dt} &= \psi_i(\beta_1^{(0)}, \quad , \beta_6^{(0)}, \gamma_1^{(0)}, \quad , \gamma_6^{(0)}, t), \\
\frac{d\gamma_i^{(1\,0\,0)}}{dt} &= \chi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \gamma_1^{(0)}, \quad , \gamma_6^{(0)}, t), \\
\frac{d\gamma_i^{(0\,1\,0)}}{dt} &= \chi_i(\beta_1^{(0)}, \quad , \beta_6^{(0)}, \gamma_1^{(0)}, \quad , \gamma_6^{(0)}, t), \\
\frac{d\gamma_i^{(0\,0\,1)}}{dt} &= 0
\end{aligned}
\right.
$$

If there were more planets more equations of the same type would be added Consider the perturbations of the first order of the elements of the orbits m_1, they are composed of two distinct parts given by the second and third equations of (28), one coming from the attraction of m and the other from the attraction of m_3 Therefore, the statement of astronomers that the perturbing effects of the various planets may be considered separately, is true for the perturbations of the first order with respect to the masses

171 The Terms of the Second Order It has been shown that $\alpha_i^{(1,\,0)} = \alpha_i^{(\;\;0)} = \beta_i^{(0\;\;1)} = \beta_i^{(0\;\;2)} = 0$, therefore it follows from (24) that the terms of the second order with respect to the masses are determined by the equations

$$
(29) \quad \left\{
\begin{aligned}
\frac{d\alpha_i^{(1\;\;1)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t)}{\partial\beta_j} \beta_j^{(1\;\;0)}, \\
\frac{d\alpha_i^{(0\;\;)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t)}{\partial\alpha_j} \alpha_j^{(0\;\;1)}, \\
\frac{d\beta_i^{(1\;\;1)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\psi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t)}{\partial\alpha_j} \alpha_j^{(0\;\;1)}, \\
\frac{d\beta_i^{(2,\,0)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\psi_i(\alpha_1^{(0)}, \quad , \alpha_6^{(0)}, \beta_1^{(0)}, \quad , \beta_6^{(0)}, t)}{\partial\beta_j} \beta_j^{(1\;\;0)}
\end{aligned}
\right.
$$

The perturbations of the first order are those which would result if the disturbing forces at every instant were the same as they would be if the bodies were moving in the original ellipses If the bodies m_1 and m_2 move in curves differing from the original ellipses the rates at which the elements will be changing at every instant will be different from the values given in (27) The perturbations of the elements of the orbit of m_1 due to the fact that m_2 departs from its original ellipse by perturbations of the first order are given by the equations of the type of the first of (29), for, if $\beta_j^{(1,0)} = 0$, it follows that $\alpha_i^{(1,1)} = 0$ also The perturbations of the elements of the orbit of m_1 due to the fact that m_1 departs from its original ellipse by perturbations of the first order are given by the equations of the type of the second of (29), for, if $\alpha_j^{(0,1)} = 0$, it follows that $\alpha_i^{(0,2)} = 0$ also The terms $\beta_i^{(1,1)}$ and $\beta_i^{(2,0)}$ in the elements of the orbit of m_2 arise from similar causes Thus the perturbations of the second order correct the errors in the terms of the first order, and those of the third order the errors in the second, and so on

As has been said, this process converges if the interval of time is taken not too great In a general way, the smaller the masses of the planets the longer the time during which the series converge In the Lunar Theory the sun plays the rôle of the disturbing planet Since this mass is very great compared to that of the central body, the earth, the series in powers of the masses as given above would converge for only a very short time, probably only a few months instead of years Such a Lunar Theory would be entirely unsatisfactory On this account the perturbations in the Lunar Theory are developed in powers of the ratio of the distances of the moon and the sun from the earth

If there is a third planet the perturbations of the second order are considerably more complicated Let the planets be m_1, m_2, and m_3, and consider the perturbations of the second order of the elements of the orbit of m_1 The following sorts of terms will arise (*a*) terms arising from the disturbing action of m_2 and m_3, due respectively to the perturbations of the first order of the elements of m_2 and m_3 by m_1, (*b*) terms arising from the disturbing action of m_2 and m_3, due to the perturbations of the first order of the elements of the orbit of m_1 by m_2 and m_3, (*c*) terms arising from the disturbing action of m_2, due to the perturbations of the first order of the elements of the orbit of m_1 by m_3, (*d*) terms arising from the disturbing action of m_2, due to the perturbations of the first order of the elements of the orbit of m_2 by m_3,

(*e*) terms arising from the disturbing action of m_3, due to the perturbations of the first order of the elements of the orbit of m_1 by m , and (*f*) terms arising from the disturbing action of m_3, due to the perturbations of the first order of the elements of m_3 by m

Under the supposition that there are three planets, the terms of the second order with respect to the masses are found from equations (19) and (20) to be

$$
(30)\quad \left\{
\begin{aligned}
\frac{d\alpha_i^{(1\,1\,0)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \beta_1^{(0)}, \ldots, \beta_6^{(0)}, t)}{\partial\beta_j}\,\beta_j^{(1\,0\,0)}, \\
\frac{d\alpha_i^{(1\,0\,1)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_6^{(0)}, t)}{\partial\gamma_j}\,\gamma_j^{(1\,0\,0)}, \\
\frac{d\alpha_i^{(0\,2\,0)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \beta_1^{(0)}, \ldots, \beta_6^{(0)}, t)}{\partial\alpha_j}\,\alpha_j^{(0\,1\,0)}, \\
\frac{d\alpha_i^{(0\,0\;)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_6^{(0)}, t)}{\partial\alpha_j}\,\alpha_j^{(0\,0\,1)}, \\
\frac{d\alpha_i^{(0\,1\,1)}}{dt} &= \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \beta_1^{(0)}, \ldots, \beta_6^{(0)}, t)}{\partial\alpha_j}\,\alpha_j^{(0\,0\,1)} \\
&+ \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \beta_1^{(0)}, \ldots, \beta_6^{(0)}, t)}{\partial\beta_j}\,\beta_j^{(0\,0\,1)} \\
&+ \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_6^{(0)}, t)}{\partial\alpha_j}\,\alpha_j^{(0\,1\,0)} \\
&+ \sum_{j=1}^{6} \frac{\partial\phi_i(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}, \gamma_1^{(0)}, \ldots, \gamma_6^{(0)}, t)}{\partial\gamma_j}\,\gamma_j^{(0\,1\,0)},
\end{aligned}
\right.
$$

and similar equations for $\frac{d\beta_i}{dt}$ and $\frac{d\gamma_i}{dt}$

The first two equations give the perturbations of the class (*a*), for, $\phi_i(\alpha, \beta)$ and $\phi_i(\alpha, \gamma)$ are the portions of the perturbative function given by m_2 and m_3 respectively, while $\beta_j^{(1\,0\,0)}$ and $\gamma_j^{(1\,0\,0)}$ are the perturbations of the first order of the elements of the orbits of m_2 and m_3 by m_1 Similarly the third and fourth equations give the perturbations of the class (*b*), the first term of the fifth equation, those of class (*c*), the second term, of class (*d*), the third term, of class (*e*), and the fourth term, of the class (*f*) It appears from this that the terms of the second order cannot be computed separately for each of the planets

XXIV PROBLEMS

1 Show that equations (7) identically satisfy (1) for $f\left(x, \frac{dx}{dt}, t\right) = \sin(kt)$ when the values of a_1 and a_2 given in (10) are used

2 The illustrative example (1) was particularly simple in that the right side was taken as a function of t alone Suppose the right side is mx, where $k=1$ and $m=0\ 1$ Suppose $x=1$, $\frac{dx}{dt}=0$, at $t=0$ Integrate by the method for linear homogeneous equations and compute the integral for $t=1$ Integrate again by the method used in computing perturbations, starting from the equations corresponding to (7), and compute the value of the integral for the same time

3 Write the equations defining the terms of order zero, one, and two, in the masses when equations (11) are integrated as series in m_1 and m_2 Show that the terms of order zero are the coordinates that m_1 and m_2 would have if they were particles moving around the sun in ellipses defined by their initial conditions Show that the equations defining the terms of the first and higher orders are linear and non homogeneous, instead of being reduced to quadratures as when the method of the variation of parameters is used

4 Suppose there were four planets, m_1, m_2, m_3, m_4, write all the terms of the second order with respect to the masses according to (30) and interpret each

5 Suppose there are two planets m_1 and m_2, write all of the terms of the third order with respect to the masses and interpret each

6 Suppose $m_1=m_2=m_3$ and that the planets are arranged in the order m_1, m_2, m_3 with respect to their distance from the sun. Show that of the perturbations defined by equations (30) the most important are those given by the first and third equations and the second term of the fifth, that the perturbations next in importance are given by the first, third, and fourth terms of the fifth equation, and that the least important are given by the second and fourth equations

172 Choice of Elements In order to exhibit the manner in which the various sorts of terms enter in the perturbations of the first order, it will be necessary to develop equations (19) explicitly This was deferred, on account of the length of the transformations which are necessary, until a general view of the mathematical principles involved could be given

If terms of the first order alone are considered the functions $\phi_i(\alpha, \beta)$ may be considered independently of $\psi_i(\alpha, \beta)$ Any independent functions of the elements may be used in place of the ordinary elements In fact, one of the elements already employed, $\pi = \omega + \Omega$, is the sum of two geometrically simpler elements Now the form of $\phi_i(\alpha, \beta)$ will depend upon the elements chosen, they will be taken in the first example which follows so that those functions shall become as simple as possible

173 Lagrange's Brackets Lagrange has made the following transformation which greatly facilitates the computation of equations (19) Multiply equations (18) by $-\frac{\partial x_1'}{\partial \alpha_1}$, $-\frac{\partial y_1'}{\partial \alpha_1}$, $-\frac{\partial z_1'}{\partial \alpha_1}$, $\frac{\partial x_1}{\partial \alpha_1}$, $\frac{\partial y_1}{\partial \alpha_1}$, $\frac{\partial z_1}{\partial \alpha_1}$ respectively and add The result is

$$(31)\quad \left\{\begin{aligned} &\frac{d\alpha}{dt}\left(\frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_1'}{\partial \alpha} - \frac{\partial x_1'}{\partial \alpha_1}\frac{\partial x_1}{\partial \alpha_2} + \frac{\partial y_1}{\partial \alpha_1}\frac{\partial y_1'}{\partial \alpha_2} - \frac{\partial y_1'}{\partial \alpha_1}\frac{\partial y_1}{\partial \alpha} + \frac{\partial z_1}{\partial \alpha_1}\frac{\partial z_1'}{\partial \alpha_2} - \frac{\partial z_1'}{\partial \alpha_1}\frac{\partial z_1}{\partial \alpha_2}\right) \\ &+ \frac{d\alpha_3}{dt}\left(\frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_1'}{\partial \alpha_3} - \frac{\partial x_1'}{\partial \alpha_1}\frac{\partial x_1}{\partial \alpha_3} + \quad\right) + \quad + \frac{d\alpha_6}{dt}\left(\frac{\partial x_1}{\partial \alpha_1}\frac{\partial x_1'}{\partial \alpha_6} - \frac{\partial x_1'}{\partial \alpha_1}\frac{\partial x_1}{\partial \alpha_6} + \quad\right) \\ &= m \frac{\partial R_{1\,2}}{\partial x_1}\frac{\partial x_1}{\partial \alpha_1} + m_2 \frac{\partial R_{1\,2}}{\partial y_1}\frac{\partial y_1}{\partial \alpha_1} + m_2 \frac{\partial R_{1\,2}}{\partial z_1}\frac{\partial z_1}{\partial \alpha_1} = m_2 \frac{\partial R_{1\,2}}{\partial \alpha_1} \end{aligned}\right.$$

Let

$$(32)\quad [\alpha_i, \alpha_j] = \frac{\partial x_1}{\partial \alpha_i}\frac{\partial x_1'}{\partial \alpha_j} - \frac{\partial x_1'}{\partial \alpha_i}\frac{\partial x_1}{\partial \alpha_j} + \frac{\partial y_1}{\partial \alpha_i}\frac{\partial y_1}{\partial \alpha_j} - \frac{\partial y_1'}{\partial \alpha_i}\frac{\partial y_1}{\partial \alpha_j} + \frac{\partial z_1}{\partial \alpha_i}\frac{\partial z_1'}{\partial \alpha_j} - \frac{\partial z_1'}{\partial \alpha_i}\frac{\partial z_1}{\partial \alpha_j},$$

and form the equations corresponding to (31) in $\alpha_2, \quad, \alpha_6$ The resulting system of equations is

$$(33)\quad \left\{\begin{aligned} &\sum_{i=1}^{6} [\alpha_1, \alpha_i] \frac{d\alpha_i}{dt} = m_2 \frac{\partial R_{1\,2}}{\partial \alpha_1}, \\ &\sum_{i=1}^{6} [\alpha_2, \alpha_i] \frac{d\alpha_i}{dt} = m_2 \frac{\partial R_{1\,2}}{\partial \alpha_2}, \\ & \\ &\sum_{i=1}^{6} [\alpha_6, \alpha_i] \frac{d\alpha_i}{dt} = m \frac{\partial R_{1\,2}}{\partial \alpha_6} \end{aligned}\right.$$

These equations are equivalent to the system (18) and will be used in place of them

174 Properties of Lagrange's Brackets It follows at once from the definitions of the brackets that

$$(34)\qquad \begin{cases} [a_i,\ a_i] = 0, \\ [a_i,\ a_j] = -[a_j,\ a_i] \end{cases}$$

A more important property is that the brackets do not contain the time explicitly, that is,

$$(35)\qquad \frac{\partial\,(a_i,\ a_j)}{\partial t} = 0, \quad (i = 1, \quad , 6,\ j = 1, \quad , 6)$$

Many complicated expressions will arise in the following which are symmetrical in x, y, and z In order to abbreviate the writing let S, standing before a function of x, indicate that the same functions of y and z are to be added Thus, for example,

$$S\,(x_1x_2' - x_2x_1') \equiv (x_1x_2' - x_2x_1') + (y_1y_2' - y_2y_1') + (z_1z_2' - z_2z_1')$$

In starting from the definitions of the brackets and omitting the subscripts of x, $\quad$, z', which will not be of use in what follows, it is found that

$$\begin{aligned}
\frac{\partial\,[a_i,\ a_j]}{\partial t} &= S\left(\frac{\partial^2 x}{\partial a_i \partial t}\frac{\partial x'}{\partial a_j} + \frac{\partial x}{\partial a_i}\frac{\partial^2 x'}{\partial a_j \partial t} - \frac{\partial^2 x'}{\partial a_i \partial t}\frac{\partial x}{\partial a_j} - \frac{\partial x}{\partial a_i}\frac{\partial^2 x}{\partial a_j \partial t}\right) \\
&= \frac{\partial}{\partial a_i} S\left(\frac{\partial x}{\partial t}\frac{\partial x'}{\partial a_j} - \frac{\partial x'}{\partial t}\frac{\partial x}{\partial a_j}\right) + S\left(-\frac{\partial x}{\partial t}\frac{\partial^2 x'}{\partial a_i \partial a_j} + \frac{\partial x'}{\partial t}\frac{\partial^2 x}{\partial a_i \partial a_j}\right) \\
&\qquad + S\left(\frac{\partial x}{\partial a_i}\frac{\partial^2 x'}{\partial a_j \partial t} - \frac{\partial x'}{\partial a_i}\frac{\partial^2 x}{\partial a_j \partial t}\right) \\
&= \frac{\partial}{\partial a_i} S\left(\frac{\partial x}{\partial t}\frac{\partial x'}{\partial a_j} - \frac{\partial x'}{\partial t}\frac{\partial x}{\partial a_j}\right) - \frac{\partial}{\partial a_j} S\left(\frac{\partial x}{\partial t}\frac{\partial x'}{\partial a_i} - \frac{\partial x'}{\partial t}\frac{\partial x}{\partial a_i}\right)
\end{aligned}$$

The partial derivatives of the coordinates with respect to the time are the same in disturbed motion as the total derivatives in undisturbed motion Therefore this equation becomes as a consequence of (14)

$$\begin{aligned}
\frac{\partial\,[a_i,\ a_j]}{\partial t} &= \frac{\partial}{\partial a_i} S\left(\frac{\partial \Omega}{\partial x}\frac{\partial x}{\partial a_j} + \frac{\partial \Omega}{\partial x'}\frac{\partial x'}{\partial a_j}\right) - \frac{\partial}{\partial a_j} S\left(\frac{\partial \Omega}{\partial x}\frac{\partial x}{\partial a_i} + \frac{\partial \Omega}{\partial x'}\frac{\partial x'}{\partial a_i}\right) \\
&= \frac{\partial}{\partial a_i}\left(\frac{\partial \Omega}{\partial a_j}\right) - \frac{\partial}{\partial a_j}\left(\frac{\partial \Omega}{\partial a_i}\right) = \frac{\partial^2 \Omega}{\partial a_i \partial a_j} - \frac{\partial^2 \Omega}{\partial a_j \partial a_i} = 0 \qquad \text{Q E D}
\end{aligned}$$

Since the brackets do not contain the time explicitly they may be computed for any epoch whatever, and in particular for $t = t_0$ The equations become very simple if the coordinates at the time $t = t_0$ are taken for the a_1, $\quad$, a_6 Let the coordinates at the time $t = t_0$ be x_0, $\quad$, z_0', then

$$[x_0,\ y_0] = S\left(\frac{\partial x_0}{\partial x_0}\frac{\partial x_0'}{\partial y_0} - \frac{\partial x_0'}{\partial x_0}\frac{\partial x_0}{\partial y_0}\right)$$

which equals zero because x_0' is independent of y_0 and x_0. Similarly

$$(36)\qquad [y_0, z_0] = [z_0, x_0] = [x_0', y_0'] = [y_0', z_0'] = [z_0', x_0'] = [x_0, y_0] = 0$$

But

$$(37)\qquad [x_0, x_0'] = [y_0, y_0'] = [z_0, z_0'] = 1$$

Therefore equations (33) become in this case

$$(38)\qquad \begin{cases} \dfrac{dx_0}{dt} = m_2 \dfrac{\partial R_{1,2}}{\partial x_0'}, & \dfrac{dx_0'}{dt} = -m_2 \dfrac{\partial R_{1,2}}{\partial x_0}, \\[2ex] \dfrac{dy_0}{dt} = m_2 \dfrac{\partial R_{1,2}}{\partial y_0'}, & \dfrac{dy_0'}{dt} = -m_2 \dfrac{\partial R_{1,2}}{\partial y_0}, \\[2ex] \dfrac{dz_0}{dt} = m_2 \dfrac{\partial R_{1,2}}{\partial z_0'}, & \dfrac{dz_0'}{dt} = -m_2 \dfrac{\partial R_{1,2}}{\partial z_0} \end{cases}$$

Any system of elements which gives equations of this form is known as a *canonical system*. If these equations were solved they would give the values of the coordinates at t_0 which would have to be used to obtain the true coordinates at the time t, in supposing that the planet had moved in an undisturbed ellipse in the interval $t - t_0$. If the variables were the elliptic elements the solutions of the equations would give the elements which would have to be used to compute the coordinates at the time t, when they are supposed to have been constant during the interval $t - t_0$. Thus, when the elements have been found the remainder of the computation is that of undisturbed motion. Since the initial coordinates define the elements, as was shown in Chapter V., the variables of this article are equivalent to the elements.

175. Transformation to the Ordinary Elements. The elements used in Astronomy are not the coordinates at $t = t_0$, but Ω, i, a, e, π, and T (or $\epsilon = \pi - nT$), which were expressed in terms of the initial conditions in Arts. 86, 87, and 88. It will be necessary, therefore, to transform equations (38) to the corresponding ones which involve only the elements which are actually in use by astronomers.

Let s represent any of the elements Ω, i, a, e, π, ϵ. It may be expressed symbolically in terms of the initial conditions by

$$(39)\qquad s = f(x_0, y_0, z_0, x_0', y_0', z_0').$$

Hence it follows that

$$\frac{ds}{dt} = S\left(\frac{\partial f}{\partial x_0}\frac{dx_0}{dt} + \frac{\partial f}{\partial x_0'}\frac{dx_0'}{dt}\right),$$

or, because of (38),

$$(40)\qquad \frac{ds}{dt} = m_2 S\left(\frac{\partial f}{\partial x_0}\frac{\partial R_{1,2}}{\partial x_0'} - \frac{\partial f}{\partial x_0'}\frac{\partial R_{1,2}}{\partial x_0}\right)$$

The partial derivatives of $R_{1\,2}$ are expressed in terms of the partial derivatives with respect to the new variables by the equations

$$
(41)\quad \left\{
\begin{aligned}
\frac{\partial R_{1\,2}}{\partial x_0} &= \frac{\partial R_{1\,2}}{\partial \Omega}\frac{\partial \Omega}{\partial x_0} + \frac{\partial R_{1\,2}}{\partial i}\frac{\partial i}{\partial x_0} + \frac{\partial R_{1\,2}}{\partial a}\frac{\partial a}{\partial x_0} + \frac{\partial R_{1\,2}}{\partial e}\frac{\partial e}{\partial x_0} \\
&\qquad + \frac{\partial R_{1\,2}}{\partial \pi}\frac{\partial \pi}{\partial x_0} + \frac{\partial R_{1\,2}}{\partial \epsilon}\frac{\partial \epsilon}{\partial x_0}, \\
\frac{\partial R_{1\,2}}{\partial z_0'} &= \frac{\partial R_{1\,2}}{\partial \Omega}\frac{\partial \Omega}{\partial z_0'} + \frac{\partial R_{1\,2}}{\partial i}\frac{\partial i}{\partial z_0'} + \frac{\partial R_{1\,2}}{\partial a}\frac{\partial a}{\partial z_0'} + \frac{\partial R_{1\,2}}{\partial e}\frac{\partial e}{\partial z_0'} \\
&\qquad + \frac{\partial R_{1\,2}}{\partial \pi}\frac{\partial \pi}{\partial z_0'} + \frac{\partial R_{1\,2}}{\partial \epsilon}\frac{\partial \epsilon}{\partial z_0}
\end{aligned}
\right.
$$

Carrying out the very complicated computations of $\frac{\partial s}{\partial x_0}, \quad , \frac{\partial s}{\partial z_0'}$ by means of the equations given in Arts 86, 87, and 88, and expressing all the partial derivatives in terms of the new variables, $\frac{\partial R_{1\,2}}{\partial x_0}, \quad , \frac{\partial R_{1\,2}}{\partial z_0'}$ are found in terms of the elements and $\frac{\partial R_{1\,2}}{\partial \Omega}, \quad , \frac{\partial R_{1\,2}}{\partial \epsilon}$ Substituting in (40) and expressing $\frac{\partial f}{\partial x_0}, \quad , \frac{\partial f}{\partial z_0'}$ in terms of the elements, $\frac{ds}{dt}$ is found in terms of the elements and the derivatives of the perturbative function, $R_{1\,2}$, with respect to the elements

176 Method of Direct Computation of Lagrange's Brackets The transformations required in the method of the preceding article are very laborious, and the direct computation of the brackets, though considerably involved, is to be preferred from a practical point of view All of the computation in the transformations of this sort might be avoided by using canonical variables, but, in order to employ them, a lengthy digression upon the properties of canonical systems would be necessary, and such a discussion is outside the limits of this work Still, the labor may be notably reduced by first taking elements somewhat different from those defined in Chapter V, and then transforming to those in more ordinary use The following is based on Tisserand's exposition of Lagrange's method*

Let xy be the ecliptic, ΩP the projection of the orbit upon the celestial sphere, Π the projection of the perihelion point, and P the projection of the position of the planet at the time t In place of π

* Tisserand's *Mécanique Céleste*, vol I p 179

and ϵ, adopt the new elements ω and σ defined by the equations

$$(42) \qquad \begin{cases} \omega = \pi - \Omega, \\ \sigma = -nT \end{cases}$$

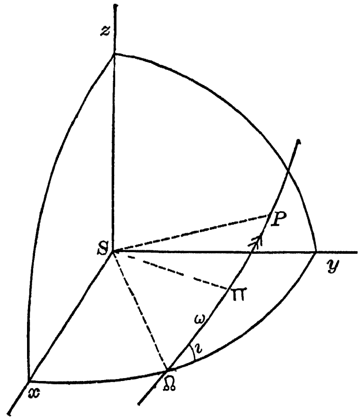

Fig 49

The following equations are either given in Art 98, or are obtained from Fig 49 by the fundamental formulas of Trigonometry

$$(43) \qquad \begin{cases} r = a\,(1 - e\cos E), \\ \tan\dfrac{v}{2} = \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E}{2}, \\ \cos v = \dfrac{\cos E - e}{1 - e\cos E}, \\ \sin v = \dfrac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \\ n = \dfrac{k\sqrt{S + m_2}}{a^{\frac{3}{2}}}, \\ E - e\sin E = nt + \sigma, \\ x = r\,\{\cos(v+\omega)\cos\Omega - \sin(v+\omega)\sin\Omega\cos i\}, \\ y = r\,\{\cos(v+\omega)\sin\Omega + \sin(v+\omega)\cos\Omega\cos i\}, \\ z = r\sin(v+\omega)\sin i \end{cases}$$

From these equations and their derivatives with respect to the time the partial derivatives of the coordinates with respect to the elements must be computed The elements have been chosen in such a manner that they are divided into two groups having distinct properties, Ω, i, and ω define the position of the plane of motion and the orientation of the orbit in the plane, and a, e, and σ define the orbit and the position of the planet in it Therefore the coordinates in the orbit may be

expressed in terms of the elements of the second group alone, and from them, the coordinates in space may be found by means of the first group alone

Take a new system of axes with the origin at the sun, the positive end of the ξ-axis directed to the perihelion point, the η-axis 90° forward in the plane of the orbit, and the ζ-axis perpendicular to the plane of the orbit Let the direction cosines between the x-axis and the ξ, η, and ζ-axes be α, α', α'', between the y-axis and the ξ, η, and ζ-axes be β, β', β'', and between the z-axis and the ξ, η, and ζ-axes be γ, γ', γ'' Then it follows from Fig 49 that

$$(44)\quad \begin{cases} \alpha = \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i, \\ \beta = \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i, \\ \gamma = \sin\omega\sin i, \\ \alpha' = -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i, \\ \beta' = -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i, \\ \gamma' = \cos\omega\sin i, \\ \alpha'' = \sin\Omega\sin i, \\ \beta'' = -\cos\Omega\sin i, \\ \gamma'' = \cos i \end{cases}$$

The following relations exist among these nine direction cosines, as can easily be verified

$$(45)\quad \begin{cases} \alpha^2+\beta^2+\gamma^2=1, & \alpha\alpha'+\beta\beta'+\gamma\gamma'=0, \\ \alpha'^2+\beta'^2+\gamma'^2=1, & \alpha'\alpha''+\beta'\beta''+\gamma'\gamma''=0, \\ \alpha''^2+\beta''^2+\gamma''^2=1, & \alpha''\alpha+\beta''\beta+\gamma''\gamma=0, \\ \alpha=\beta'\gamma''-\gamma'\beta'', \quad \alpha'=\beta''\gamma-\gamma''\beta, & \alpha''=\beta\gamma'-\gamma\beta', \\ \beta=\gamma'\alpha''-\alpha'\gamma'', \quad \beta'=\gamma''\alpha-\alpha''\gamma, & \beta''=\gamma\alpha'-\alpha\gamma', \\ \gamma=\alpha'\beta''-\beta'\alpha'', \quad \gamma'=\alpha''\beta-\beta''\alpha, & \gamma''=\alpha\beta'-\beta\alpha' \end{cases}$$

It follows from (43) and (44) and the definition of the new system of axes that

$$(46)\quad \begin{cases} \xi = r\cos v = a(\cos E - e), \quad \eta = a\sqrt{1-e^2}\sin E, \\ \dfrac{\partial E}{\partial t} = \dfrac{n}{1-e\cos E}, \quad \xi' = \dfrac{-na\sin E}{1-e\cos E} = \dfrac{-k\sqrt{S+m_2}\sin E}{\sqrt{a}(1-e\cos E)}, \\ \eta' = \dfrac{na\sqrt{1-e^2}\cos E}{1-e\cos E} = \dfrac{k\sqrt{S+m_2}\sqrt{1-e^2}\cos E}{\sqrt{a}(1-e\cos E)}, \\ x = \alpha\xi+\alpha'\eta, \quad y = \beta\xi+\beta'\eta, \quad z = \gamma\xi+\gamma'\eta, \\ x' = \alpha\xi'+\alpha'\eta', \quad y' = \beta\xi'+\beta'\eta', \quad z' = \gamma\xi'+\gamma'\eta' \end{cases}$$

The partial derivatives of α, , γ'' with respect to the elements may be computed once for all, they are found from (44) to be

$$(47)\quad \begin{cases} \dfrac{\partial\alpha}{\partial\omega}=\alpha, & \dfrac{\partial\alpha'}{\partial\omega}=-\alpha, & \dfrac{\partial\alpha''}{\partial\omega}=0, \\ \dfrac{\partial\beta}{\partial\omega}=\beta', & \dfrac{\partial\beta'}{\partial\omega}=-\beta, & \dfrac{\partial\beta''}{\partial\omega}=0, \\ \dfrac{\partial\gamma}{\partial\omega}=\gamma', & \dfrac{\partial\gamma'}{\partial\omega}=-\gamma, & \dfrac{\partial\gamma''}{\partial\omega}=0 \end{cases}$$

$$(48)\quad \begin{cases} \dfrac{\partial\alpha}{\partial\Omega}=-\beta, & \dfrac{\partial\alpha'}{\partial\Omega}=-\beta', & \dfrac{\partial\alpha''}{\partial\Omega}=-\beta'', \\ \dfrac{\partial\beta}{\partial\Omega}=\alpha, & \dfrac{\partial\beta'}{\partial\Omega}=\alpha', & \dfrac{\partial\beta''}{\partial\Omega}=\alpha'', \\ \dfrac{\partial\gamma}{\partial\Omega}=0, & \dfrac{\partial\gamma'}{\partial\Omega}=0, & \dfrac{\partial\gamma'}{\partial\Omega}=0 \end{cases}$$

$$(49)\quad \begin{cases} \dfrac{\partial\alpha}{\partial i}=\alpha''\sin\omega, & \dfrac{\partial\alpha'}{\partial i}=\alpha''\cos\omega, & \dfrac{\partial\alpha'}{\partial i}=\sin\Omega\cos i, \\ \dfrac{\partial\beta}{\partial i}=\beta'\sin\omega, & \dfrac{\partial\beta'}{\partial i}=\beta''\cos\omega, & \dfrac{\partial\beta'}{\partial i}=-\cos\Omega\cos i, \\ \dfrac{\partial\gamma}{\partial i}=\gamma''\sin\omega, & \dfrac{\partial\gamma'}{\partial i}=\gamma'\cos\omega, & \dfrac{\partial\gamma''}{\partial i}=-\sin i \end{cases}$$

There are as many brackets to be computed as there are combinations of the six elements taken two at a time, or $\dfrac{6!}{2!\,4!}=15$ Three of them involve elements of only the first group, nine, one element of the first group and one of the second, and three, elements of only the second group Let K and L represent any of the elements of the first group, Ω, i, ω, and P and Q any of the elements of the second group, a, e, σ Then the Lagrangian brackets to be computed are

$$(50)\quad \begin{cases} (a)\quad [K, L]=S\left(\dfrac{\partial x}{\partial K}\dfrac{\partial x'}{\partial L}-\dfrac{\partial x}{\partial K}\dfrac{\partial x}{\partial L}\right), \text{ (3 equations)}, \\ (b)\quad [K, P]=S\left(\dfrac{\partial x}{\partial K}\dfrac{\partial x'}{\partial P}-\dfrac{\partial x'}{\partial K}\dfrac{\partial x}{\partial P}\right), \text{ (9 equations)}, \\ (c)\quad [P, Q]=S\left(\dfrac{\partial x}{\partial P}\dfrac{\partial x'}{\partial Q}-\dfrac{\partial x}{\partial P}\dfrac{\partial x}{\partial Q}\right), \text{ (3 equations)} \end{cases}$$

It is found from (46) that

$$(51)\quad \begin{cases} \dfrac{\partial x}{\partial K}=\xi\dfrac{\partial\alpha}{\partial K}+\eta\dfrac{\partial\alpha}{\partial K}, & \dfrac{\partial x'}{\partial K}=\xi'\dfrac{\partial\alpha}{\partial K}+\eta'\dfrac{\partial\alpha'}{\partial K}, \\ \dfrac{\partial x}{\partial P}=\alpha\dfrac{\partial\xi}{\partial P}+\alpha'\dfrac{\partial\eta}{\partial P}, & \dfrac{\partial x'}{\partial P}=\alpha\dfrac{\partial\xi'}{\partial P}+\alpha'\dfrac{\partial\eta'}{\partial P}, \end{cases}$$

and similar equations in y and z

177 Computation of $[\omega, \Omega]$, $[\Omega, i]$, $[i, \omega]$ Let S indicate that the sum of the functions, symmetrical in α, β, and γ, is to be taken Then, the first equation of (50) becomes as a consequence of (51)

$$[K, L] = (\xi\eta' - \eta\xi')\, S\left(\frac{\partial \alpha}{\partial K}\frac{\partial \alpha'}{\partial L} - \frac{\partial \alpha'}{\partial K}\frac{\partial \alpha}{\partial L}\right)$$

But the law of areas gives

$$\xi\eta' - \eta\xi' = \xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} = k\sqrt{(S+m_2)\, a\,(1-e^2)} = na^2\sqrt{1-e^2}$$

Therefore

$$(52) \qquad [K, L] = na^2\sqrt{1-e^2}\, S\left(\frac{\partial \alpha}{\partial K}\frac{\partial \alpha'}{\partial L} - \frac{\partial \alpha'}{\partial K}\frac{\partial \alpha}{\partial L}\right)$$

Computing the right member of this equation by means of (47), (48), and (49), and reducing by means of (45), the brackets involving elements of only the first group are found to be

$$(53) \quad \begin{cases} [\omega, \Omega] = na^2\sqrt{1-e^2}\,(-\alpha\beta - \alpha'\beta' + \alpha\beta + \alpha\,\beta') = 0, \\ [\Omega, i] = na^2\sqrt{1-e^2}\,\{(\alpha\beta'' - \beta\alpha'')\cos\omega + (\beta'\alpha'' - \alpha'\beta'')\sin\omega\} \\ \qquad\quad = na^2\sqrt{1-e^2}\,(-\gamma'\cos\omega - \gamma\sin\omega) = -na^2\sqrt{1-e^2}\sin i, \\ [i, \omega] = -na^2\sqrt{1-e^2}\,\{(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'')\cos\omega + (\alpha''\alpha + \beta''\beta + \gamma''\gamma)\sin\omega\} = 0 \end{cases}$$

178 Computation of $[K, P]$ The second equation of (50) becomes, as a consequence of (51),

$$[K, P] = S\left\{\left(\xi\frac{\partial \alpha}{\partial K} + \eta\frac{\partial \alpha'}{\partial K}\right)\left(\alpha\frac{\partial \xi'}{\partial P} + \alpha'\frac{\partial \eta'}{\partial P}\right) - \left(\xi'\frac{\partial \alpha}{\partial K} + \eta'\frac{\partial \alpha'}{\partial K}\right)\left(\alpha\frac{\partial \xi}{\partial P} + \alpha'\frac{\partial \eta}{\partial P}\right)\right\}$$

$$= \left(\alpha\frac{\partial \alpha}{\partial K} + \beta\frac{\partial \beta}{\partial K} + \gamma\frac{\partial \gamma}{\partial K}\right)\left(\xi\frac{\partial \xi'}{\partial P} - \xi'\frac{\partial \xi}{\partial P}\right)$$

$$+ \left(\alpha'\frac{\partial \alpha'}{\partial K} + \beta'\frac{\partial \beta'}{\partial K} + \gamma'\frac{\partial \gamma'}{\partial K}\right)\left(\eta\frac{\partial \eta'}{\partial P} - \eta'\frac{\partial \eta}{\partial P}\right)$$

$$+ \left(\alpha\frac{\partial \alpha'}{\partial K} + \beta\frac{\partial \beta'}{\partial K} + \gamma\frac{\partial \gamma'}{\partial K}\right)\left(\eta\frac{\partial \xi'}{\partial P} - \eta'\frac{\partial \xi}{\partial P}\right)$$

$$+ \left(\alpha'\frac{\partial \alpha}{\partial K} + \beta'\frac{\partial \beta}{\partial K} + \gamma'\frac{\partial \gamma}{\partial K}\right)\left(\xi\frac{\partial \eta'}{\partial P} - \xi'\frac{\partial \eta}{\partial P}\right)$$

It follows from equations (45), (47), (48), and (49) that

$$\alpha\frac{\partial\alpha}{\partial K}+\beta\frac{\partial\beta}{\partial K}+\gamma\frac{\partial\gamma}{\partial K}=0,$$

$$\alpha'\frac{\partial\alpha'}{\partial K}+\beta'\frac{\partial\beta'}{\partial K}+\gamma'\frac{\partial\gamma'}{\partial K}=0,$$

$$\alpha\frac{\partial\alpha'}{\partial K}+\beta\frac{\partial\beta'}{\partial K}+\gamma\frac{\partial\gamma'}{\partial K}=-\left(\alpha'\frac{\partial\alpha}{\partial K}+\beta'\frac{\partial\beta}{\partial K}+\gamma'\frac{\partial\gamma}{\partial K}\right)$$

Therefore

$$(54)\left\{\begin{aligned}[K,\ P]&=\left(\alpha'\frac{\partial\alpha}{\partial K}+\beta'\frac{\partial\beta}{\partial K}+\gamma'\frac{\partial\gamma}{\partial K}\right)\left(\xi\frac{\partial\eta'}{\partial P}+\eta'\frac{\partial\xi}{\partial P}-\xi'\frac{\partial\eta}{\partial P}-\eta\frac{\partial\xi'}{\partial P}\right)\\ &=\left(\alpha'\frac{\partial\alpha}{\partial K}+\beta'\frac{\partial\beta}{\partial K}+\gamma'\frac{\partial\gamma}{\partial K}\right)\frac{\partial(\xi\eta-\eta\xi')}{\partial P}\\ &=k\sqrt{S+m_2}\left(\alpha'\frac{\partial\alpha}{\partial K}+\beta'\frac{\partial\beta}{\partial K}+\gamma'\frac{\partial\gamma}{\partial K}\right)\frac{\partial\sqrt{a(1-e^2)}}{\partial P}\end{aligned}\right.$$

Let $P=a,\ e,\ \sigma$ in succession Then

$$(55)\left\{\begin{aligned}&k\sqrt{S+m_2}\frac{\partial\sqrt{a(1-e^2)}}{\partial a}=\frac{na}{2}\sqrt{1-e^2},\\ &k\sqrt{S+m_2}\frac{\partial\sqrt{a(1-e^2)}}{\partial e}=-\frac{na^2e}{\sqrt{1-e^2}},\\ &k\sqrt{S+m_2}\frac{\partial\sqrt{a(1-e^2)}}{\partial\sigma}=0\end{aligned}\right.$$

Let $K=\omega,\ \Omega,\ \iota$ in turn in (54), and make use of (55), then it is found that

$$(56)\left\{\begin{array}{lll}[\omega,\ a]=\dfrac{na}{2}\sqrt{1-e^2}, & [\Omega,\ a]=\dfrac{na}{2}\sqrt{1-e^2}\cos\iota, & [\iota,\ a]=0,\\ [\omega,\ e]=\dfrac{-na\,e}{\sqrt{1-e^2}}, & [\Omega,\ e]=\dfrac{-na^2e}{\sqrt{1-e^2}}\cos\iota, & [\iota,\ e]=0,\\ [\omega,\ \sigma]=0, & [\Omega,\ \sigma]=0, & [\iota,\ \sigma]=0\end{array}\right.$$

179 Computation of $[a,e]$, $[e,\sigma]$, $[\sigma,a]$ The third equation of (50) becomes, as a consequence of (51),

$$\begin{aligned}[P,\ Q]=S\Big\{&\left(\alpha\frac{\partial\xi}{\partial P}+\alpha'\frac{\partial\eta}{\partial P}\right)\left(\alpha\frac{\partial\xi'}{\partial Q}+\alpha'\frac{\partial\eta'}{\partial Q}\right)\\ &-\left(\alpha\frac{\partial\xi'}{\partial P}+\alpha'\frac{\partial\eta'}{\partial P}\right)\left(\alpha\frac{\partial\xi}{\partial Q}+\alpha'\frac{\partial\eta}{\partial Q}\right)\Big\}\\ =(\alpha+\beta+\gamma^2)&\left(\frac{\partial\xi}{\partial P}\frac{\partial\xi}{\partial Q}-\frac{\partial\xi}{\partial Q}\frac{\partial\xi'}{\partial P}\right)+(\alpha'^2+\beta'^2+\gamma'^2)\left(\frac{\partial\eta}{\partial P}\frac{\partial\eta'}{\partial Q}-\frac{\partial\eta}{\partial Q}\frac{\partial\eta'}{\partial P}\right)\\ +(\alpha\alpha+\beta\beta+\gamma\gamma')&\left(\frac{\partial\xi}{\partial P}\frac{\partial\eta'}{\partial Q}-\frac{\partial\xi}{\partial Q}\frac{\partial\eta'}{\partial P}+\frac{\partial\xi'}{\partial Q}\frac{\partial\eta}{\partial P}-\frac{\partial\xi'}{\partial P}\frac{\partial\eta}{\partial Q}\right)\end{aligned}$$

As a consequence of (45), this equation reduces to

$$(57)\qquad [P,\, Q]=\frac{\partial\xi}{\partial P}\frac{\partial\xi'}{\partial Q}-\frac{\partial\xi}{\partial Q}\frac{\partial\xi'}{\partial P}+\frac{\partial\eta}{\partial P}\frac{\partial\eta'}{\partial Q}-\frac{\partial\eta}{\partial Q}\frac{\partial\eta'}{\partial P}$$

Since the brackets do not contain the time explicitly t may be given any value after the partial derivatives have been formed The partial derivatives become the simplest when $t=T$, the time of perihelion passage For this value of t, $E=0$, $r=a(1-e)$ Then it is found from equations (46) that*

$$(58)\qquad \begin{cases} \dfrac{\partial\xi}{\partial a}=1-e, & \dfrac{\partial\eta}{\partial a}=0, \quad \dfrac{\partial\xi'}{\partial a}=0, \quad \dfrac{\partial\eta'}{\partial a}=-\dfrac{n}{2}\sqrt{\dfrac{1+e}{1-e}}, \\ \dfrac{\partial\xi}{\partial e}=-a, & \dfrac{\partial\eta}{\partial e}=0, \quad \dfrac{\partial\xi'}{\partial e}=0, \quad \dfrac{\partial\eta'}{\partial e}=\dfrac{1}{1-e}\,\dfrac{na}{\sqrt{1-e^2}}, \\ \dfrac{\partial\xi}{\partial\sigma}=0, & \dfrac{\partial\eta}{\partial\sigma}=\dfrac{\partial\eta}{\partial E}\dfrac{\partial E}{\partial\sigma}=a\sqrt{\dfrac{1+e}{1-e}}, \quad \dfrac{\partial\xi'}{\partial\sigma}=\dfrac{-na}{(1-e)^2}, \quad \dfrac{\partial\eta'}{\partial\sigma}=0 \end{cases}$$

Then equation (57) gives

$$(59)\qquad [a,\, e]=0, \quad [e,\, \sigma]=0, \quad [\sigma,\, a]=\frac{na}{2}$$

Remembering that $[a_i,\, a_j]=-[a_j,\, a_i]$, equations (33) become as a consequence of (53), (56), and (59)

$$(60)\qquad \begin{cases} \dfrac{na}{2}\sqrt{1-e^2}\,\dfrac{da}{dt}-\dfrac{na^2e}{\sqrt{1-e^2}}\dfrac{de}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial\omega}, \\ -na^2\sqrt{1-e^2}\sin i\,\dfrac{di}{dt}+\dfrac{na}{2}\sqrt{1-e^2}\cos i\,\dfrac{da}{dt}-\dfrac{na^2e}{\sqrt{1-e^2}}\cos i\,\dfrac{de}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial\Omega}, \\ na^2\sqrt{1-e^2}\sin i\,\dfrac{d\Omega}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial i}, \\ -\dfrac{na}{2}\sqrt{1-e^2}\,\dfrac{d\omega}{dt}-\dfrac{na}{2}\sqrt{1-e^2}\cos i\,\dfrac{d\Omega}{dt}-\dfrac{na}{2}\dfrac{d\sigma}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial a}, \\ \dfrac{na^2e}{\sqrt{1-e^2}}\dfrac{d\omega}{dt}+\dfrac{na^2e\cos i}{\sqrt{1-e^2}}\dfrac{d\Omega}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial e}, \\ \dfrac{na}{2}\dfrac{da}{dt}=m_2\dfrac{\partial R_{1\,2}}{\partial\sigma} \end{cases}$$

* It should be remembered that a and e enter explicitly and also implicitly through E and n, for E is defined by the equation

$$E-e\sin E=n(t-T)=\frac{k\sqrt{S+m_2}}{a^{\frac{3}{2}}}(t-T)$$

Then, e g $\dfrac{\partial\xi}{\partial a}=\cos E-e-a\sin E\,\dfrac{\partial E}{\partial a}=1-e$ when $t=T$, etc

These equations are easily solved, and give

$$
(61)\quad \left\{
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{m_2}{na\sqrt{1-e^2}\sin i}\,\frac{\partial R_{1,2}}{\partial i}, \\
\frac{di}{dt} &= \frac{m_2\cos i}{na^2\sqrt{1-e^2}\sin i}\,\frac{\partial R_{1,2}}{\partial \omega} - \frac{m_2}{na^2\sqrt{1-e^2}\sin i}\,\frac{\partial R_{1,2}}{\partial \Omega}, \\
\frac{d\omega}{dt} &= \frac{-m_2\cos i}{na^2\sqrt{1-e^2}\sin i}\,\frac{\partial R_{1,2}}{\partial i} + \frac{m_2\sqrt{1-e^2}}{na\,e}\,\frac{\partial R_{1,2}}{\partial e}, \\
\frac{da}{dt} &= \frac{2m_2}{na}\,\frac{\partial R_{1,2}}{\partial \sigma}, \\
\frac{de}{dt} &= \frac{m_2(1-e^2)}{na^2e}\,\frac{\partial R_{1,2}}{\partial \sigma} - \frac{m_2\sqrt{1-e^2}}{na^2e}\,\frac{\partial R_{1,2}}{\partial \omega}, \\
\frac{d\sigma}{dt} &= -\frac{m_2(1-e^2)}{na^2e}\,\frac{\partial R_{1,2}}{\partial e} - \frac{2m_2}{na}\,\frac{\partial R_{1,2}}{\partial a}.
\end{aligned}
\right.
$$

180. Computation of σ. The right member of the last equation of (61) contains $\frac{\partial R_{1,2}}{\partial a}$, which is the source of some practical difficulty from the fact that a enters in $R_{1,2}$ both explicitly and also implicitly through n. Let the partial differentiation of a function with respect to a variable so far as it occurs explicitly be expressed by placing the indicated operation in parenthesis. Then the last equation of (61) may be written

$$
(62)\qquad \frac{d\sigma}{dt} = \frac{-m_2(1-e^2)}{na^2e}\,\frac{\partial R_{1,2}}{\partial e} - \frac{2m_2}{na}\left(\frac{\partial R_{1,2}}{\partial a}\right) - \frac{2m_2}{na}\,\frac{\partial R_{1,2}}{\partial n}\,\frac{\partial n}{\partial a}.
$$

The mean motion n enters in the determination of the coordinates, and therefore in $R_{1,2}$, in the form

$$
n(t-T) = nt + \sigma.
$$

Hence it follows that

$$
(63)\qquad \frac{\partial R_{1,2}}{\partial n} = t\,\frac{\partial R_{1,2}}{\partial \sigma}.
$$

The mean motion is defined by the equation

$$
n = \frac{k\sqrt{S+m_2}}{a^{\frac{3}{2}}},
$$

whence

$$
(64)\qquad \left\{
\begin{aligned}
\frac{\partial n}{\partial a} &= -\frac{3}{2}\,\frac{n}{a}, \\
\frac{da}{dt} &= -\frac{2}{3}\,\frac{a}{n}\,\frac{dn}{dt}.
\end{aligned}
\right.
$$

Equation (62) becomes as a consequence of (63), (64), and the fourth of (61)

$$\frac{d\sigma}{dt} = -\frac{m_2(1-e^2)}{na^2e}\frac{\partial R_{1\,2}}{\partial e} - \frac{2m_2}{na}\left(\frac{\partial R_{1\,2}}{\partial a}\right) - t\frac{dn}{dt}$$

In order to facilitate the computation, define a new variable σ' by the equation

$$(65) \qquad \frac{d\sigma'}{dt} = -\frac{m_2(1-e^2)}{na^2e}\frac{\partial R_{1\,2}}{\partial e} - \frac{2m_2}{na}\left(\frac{\partial R_{1\,2}}{\partial a}\right)$$

Then it is found that

$$\frac{d\sigma}{dt} = \frac{d\sigma'}{dt} - t\frac{dn}{dt},$$

whence, integrating the last term by parts,

$$\sigma = \sigma' - nt + \int n dt$$

From the definition of n it follows that

$$(66) \qquad \sigma = \sigma' - nt + k\sqrt{S+m_2}\int\frac{dt}{a^{\frac{3}{2}}}$$

Then, in computing the perturbations, the first five equations of (61), and (65) and (66), are to be used To get the perturbations of the first order the first five equations of (61) and equation (65) are integrated using the osculating values of the elements in the right members The fourth equation gives

$$a = a^{(0)} + a^{(0\,1)}m_2 + \quad ,$$

whence

$$\sigma = \sigma' - nt + \frac{k\sqrt{S+m_2}}{(a^{(0)})^{\frac{3}{2}}}\int\left\{1 - \frac{3}{2}\frac{a^{(0\,1)}}{a^{(0)}}m_2 + \quad\right\}dt$$

Then the term of the first order in σ is

$$m_2\sigma^{(0\,1)} = m_2\sigma'^{(0\,1)} - \frac{k\sqrt{S+m_2}}{(a^{(0)})^{\frac{3}{2}}}\left(1 - \frac{3}{2}\frac{a^{(0\,1)}}{a^{(0)}}m_2\right)(t-t_0) + \frac{k\sqrt{S+m_2}}{(a^{(0)})^{\frac{3}{2}}}(t-t_0)$$
$$-\frac{3m_2k\sqrt{S+m_2}}{2(a^{(0)})^{\frac{5}{2}}}\int a^{(0,1)}dt,$$

whence

$$(67) \qquad \sigma^{(0\,1)} = \sigma'^{(0\,1)} + \frac{3k\sqrt{S+m_2}}{2(a^{(0)})^{\frac{5}{2}}}\{a^{(0\,1)}(t-t_0) - \int a^{(0,1)}dt\}$$

Instead of proceeding in this manner Leverrier adopted σ' as a new variable instead of σ*, replacing $\sigma + nt$ by $\sigma' + \int n dt$ Since $\frac{\partial R_{1\,2}}{\partial\sigma} = \frac{\partial R_{1\,2}}{\partial\sigma'}$

* *Annales de l'Observatoire de Paris*, vol I p 255

equations (64) keep the same form, and $R_{1\,2}$ is to be regarded as not depending upon a implicitly through n In this plan the integral $\int n dt$ is to be computed term by term with the different orders of the elements This method will be adopted here, but, for simplicity, the accent will be omitted from σ and the parenthesis from $\frac{\partial R_{1\,2}}{\partial a}$

181 Change from Ω, ω, and σ to Ω, π, and ϵ This transformation is particularly simple because the relations between the old elements ω and σ and the new ones π and ϵ are very simple They are

$$(68)\qquad \begin{cases} \Omega = \Omega, \\ \omega = \pi - \Omega, \\ \sigma = \epsilon - \pi, \end{cases}$$

whence

$$(69)\qquad \begin{cases} \dfrac{d\Omega}{dt} = \dfrac{d\Omega}{dt}, \\ \dfrac{d\omega}{dt} = \dfrac{d\pi}{dt} - \dfrac{d\Omega}{dt}, \\ \dfrac{d\sigma}{dt} = \dfrac{d\epsilon}{dt} - \dfrac{d\pi}{dt} \end{cases}$$

From equations (68) it follows that

$$(70)\qquad \begin{cases} \Omega = \Omega, \\ \pi = \omega + \Omega, \\ \epsilon = \sigma + \pi = \sigma + \omega + \Omega \end{cases}$$

Hence the transformations in the partial derivatives are given by the equations

$$(71)\qquad \begin{cases} \dfrac{\partial R_{1\,2}}{\partial \Omega} = \left(\dfrac{\partial R_{1\,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \Omega} + \left(\dfrac{\partial R_{1\,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \Omega} + \left(\dfrac{\partial R_{1\,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \Omega} \\ \qquad\quad = \left(\dfrac{\partial R_{1\,2}}{\partial \Omega}\right) + \left(\dfrac{\partial R_{1\,2}}{\partial \pi}\right) + \left(\dfrac{\partial R_{1\,2}}{\partial \epsilon}\right), \\ \dfrac{\partial R_{1\,2}}{\partial \omega} = \left(\dfrac{\partial R_{1\,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \omega} + \left(\dfrac{\partial R_{1\,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \omega} + \left(\dfrac{\partial R_{1\,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \omega} = \left(\dfrac{\partial R_{1\,2}}{\partial \pi}\right) + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right), \\ \dfrac{\partial R_{1\,2}}{\partial \sigma} = \left(\dfrac{\partial R_{1\,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \sigma} + \left(\dfrac{\partial R_{1\,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \sigma} + \left(\dfrac{\partial R_{1\,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \sigma} = \left(\dfrac{\partial R_{1\,2}}{\partial \epsilon}\right) \end{cases}$$

Substituting (69) and (71) in (61) and omitting the parentheses around the partial derivatives, it is found on solving that

$$
(72)\quad \left\{
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{m_2}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R_{1\,2}}{\partial i},\\
\frac{di}{dt} &= \frac{-m_2}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R_{1\,2}}{\partial \Omega} - \frac{m_2\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\left(\frac{\partial R_{1\,2}}{\partial \pi}+\frac{\partial R_{1\,2}}{\partial \epsilon}\right),\\
\frac{d\pi}{dt} &= \frac{m_2\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R_{1\,2}}{\partial i} + \frac{m_2\sqrt{1-e^2}}{na^2 e}\frac{\partial R_{1\,2}}{\partial e},\\
\frac{da}{dt} &= \frac{2m_2}{na}\frac{\partial R_{1\,2}}{\partial \epsilon},\\
\frac{de}{dt} &= -m_2\sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R_{1\,2}}{\partial \epsilon} - \frac{m_2\sqrt{1-e^2}}{na^2 e}\frac{\partial R_{1\,2}}{\partial \pi},\\
\frac{d\epsilon}{dt} &= \frac{m_2\tan\frac{i}{2}}{na^2\sqrt{1-e^2}}\frac{\partial R_{1\,2}}{\partial i} + m_2\sqrt{1-e^2}\,\frac{1-\sqrt{1-e^2}}{na^2 e}\frac{\partial R_{1\,2}}{\partial e}\\
&\qquad - \frac{2m_2}{na}\frac{\partial R_{1\,2}}{\partial a}
\end{aligned}
\right.
$$

These equations*, together with the corresponding ones for the elements of the planet m_2, constitute a rigorous system of differential equations for the determination of the motion of the planets m_1 and m_2 with respect to the sun when there are no other forces than the mutual attractions of the three bodies

The partial derivative, $\frac{\partial R_{1\,2}}{\partial a}$, occurs in the right member of the last equation, and it will be convenient to introduce, as in Art 180, a new variable such that the derivative of $R_{1\,2}$ with respect to a need be taken only so far as this element enters explicitly If ϵ' be defined by the equation

$$\frac{d\epsilon'}{dt} = \frac{d\epsilon}{dt} + t\frac{dn}{dt},$$

it will be found precisely as in Art 180 that this requirement is fulfilled

If $R_{1,2}$ is expressed in terms of the osculating elements at the epoch t_0 and the time, equations (72) become the explicit expressions for the

* The subscript 1, which was omitted from the coördinates and elements in Art 174, should be replaced when the equations for more than one planet are written

first half of system (27), and give the perturbations of the elements which are of the first order with respect to the masses

182 Introduction of Rectangular Components of the Disturbing Acceleration Equations (72) require for their application that $R_{1\,2}$ shall be expressed first in terms of the elements, after which the partial derivatives must be formed In some cases, especially in the orbits of comets, it is advantageous to have the rates of variation of the elements expressed in terms of three rectangular components of the disturbing acceleration

The disturbing acceleration will be resolved into three rectangular components W, S, R W is the component of acceleration acting perpendicular to the plane of the orbit, and will be taken positive when directed toward the north pole of the orbit, S is the component in the plane of the orbit which acts at right angles to the radius vector, and will be taken positive when acting in the direction of motion, R is the component acting along the radius vector, and will be taken positive when directed from the sun The components used in the last chapter evidently might be employed here instead of these, but the resulting equations would be less simple

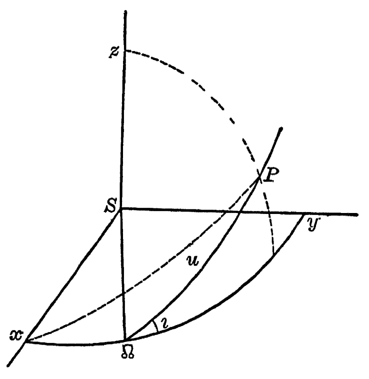

Fig 50

In order to obtain the desired equations it is only necessary to express the partial derivatives of $R_{1\,2}$ with respect to the elements in terms of W, S, and R, and to substitute them in (61) or (72), depending upon the set of elements used The transformation will be made for the elements used in equations (61)

$m_2 \frac{\partial R_{12}}{\partial x}$, $m_2 \frac{\partial R_{12}}{\partial y}$, $m_2 \frac{\partial R_{12}}{\partial z}$ are the components of the disturbing acceleration parallel to the fixed axes of reference It follows from the elementary properties of the resolution and composition of accelerations that $m_2 \frac{\partial R_{12}}{\partial x}$ is equal to the sum of the projections of W, S, and R upon the x-axis, and similarly for the others

Let u represent the argument of the latitude, or the distance from the ascending node to the planet P Then it follows from the fundamental formulas of Trigonometry that

$$(73)\quad \begin{cases} m_2 \dfrac{\partial R_{12}}{\partial x} = R(\cos u \cos \Omega - \sin u \sin \Omega \cos i) \\ \qquad\qquad - S(\sin u \cos \Omega + \cos u \sin \Omega \cos i) + W \sin \Omega \sin i, \\ m_2 \dfrac{\partial R_{12}}{\partial y} = R(\cos u \sin \Omega + \sin u \cos \Omega \cos i) \\ \qquad\qquad - S(\sin u \sin \Omega - \cos u \cos \Omega \cos i) - W \cos \Omega \sin i, \\ m_2 \dfrac{\partial R_{12}}{\partial z} = R \sin u \sin i + S \cos u \sin i + W \cos i \end{cases}$$

Let s represent any of the elements $\Omega, \ \ , \sigma$, then

$$(74)\qquad \frac{\partial R_{12}}{\partial s} = \frac{\partial R_{12}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial R_{12}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial R_{12}}{\partial z}\frac{\partial z}{\partial s}$$

The derivatives $\frac{\partial R_{12}}{\partial x}$, $\frac{\partial R_{12}}{\partial y}$, $\frac{\partial R_{12}}{\partial z}$ are given in (73) and when $\frac{\partial x}{\partial s}$, $\frac{\partial y}{\partial s}$, and $\frac{\partial z}{\partial s}$ have been found, the transformation can be completed at once

By equations (51)

$$(75)\quad \begin{cases} \dfrac{\partial x}{\partial K} = \xi \dfrac{\partial \alpha}{\partial K} + \eta \dfrac{\partial \alpha'}{\partial K}, & \dfrac{\partial x}{\partial P} = \alpha \dfrac{\partial \xi}{\partial P} + \alpha' \dfrac{\partial \eta}{\partial P}, \\ \dfrac{\partial y}{\partial K} = \xi \dfrac{\partial \beta}{\partial K} + \eta \dfrac{\partial \beta'}{\partial K}, & \dfrac{\partial y}{\partial P} = \beta \dfrac{\partial \xi}{\partial P} + \beta' \dfrac{\partial \eta}{\partial P}, \\ \dfrac{\partial z}{\partial K} = \xi \dfrac{\partial \gamma}{\partial K} + \eta \dfrac{\partial \gamma'}{\partial K}, & \dfrac{\partial z}{\partial P} = \gamma \dfrac{\partial \xi}{\partial P} + \gamma' \dfrac{\partial \eta}{\partial P}, \end{cases}$$

where K is any of the elements Ω, i, ω, and P any of the elements a, e, σ The quantities α, $\ \ $, γ' are defined in (44) and their derivatives are given in (47), (48), and (49), the derivatives $\frac{\partial \xi}{\partial P}$ and $\frac{\partial \eta}{\partial P}$ are to be computed from (46)

It is found after some rather long but simple reductions that

$$
(76)\quad \begin{cases}
m_2 \dfrac{\partial R_{1\,2}}{\partial \Omega} = Sr\cos i - Wr\cos u \sin i, \\[2ex]
m\ \dfrac{\partial R_{1\,2}}{\partial i} = Wr \sin u, \\[2ex]
m\ \dfrac{\partial R_{1\,2}}{\partial \omega} = Sr, \\[2ex]
m\ \dfrac{\partial R_{1\,2}}{\partial a} = R\dfrac{r}{a},^{*} \\[2ex]
m_2 \dfrac{\partial R_{1\,2}}{\partial e} = -Ra\cos v + S\left(1 + \dfrac{r}{p}\right) a \sin v, \\[2ex]
m_2 \dfrac{\partial R_{1\,2}}{\partial \sigma} = \dfrac{Rae}{\sqrt{1-e^2}} \sin v + S\dfrac{a}{r}\sqrt{1-e^2}
\end{cases}
$$

Therefore equations (61) become

$$
(77)\quad \begin{cases}
\dfrac{d\Omega}{dt} = \dfrac{r \sin u}{na^2\sqrt{1-e}\ \sin i} W, \\[2ex]
\dfrac{di}{dt} = \dfrac{r\cos u}{na\ \sqrt{1-e}} W, \\[2ex]
\dfrac{d\omega}{dt} = \dfrac{-\sqrt{1-e}\ \cos v}{nae} R + \dfrac{\sqrt{1-e^2}}{nae}\left(1 + \dfrac{r}{p}\right) \sin v S - \dfrac{r \sin u \cot i}{na\ \sqrt{1-e^2}} W, \\[2ex]
\dfrac{da}{dt} = \dfrac{2e\sin v}{n\sqrt{1-e}} R + \dfrac{2a\sqrt{1-e}}{nr} S, \\[2ex]
\dfrac{de}{dt} = \dfrac{\sqrt{1-e^2}\sin v}{na} R + \dfrac{\sqrt{1-e}}{na^2 e}\left\{\dfrac{a^2(1-e^2)}{r} - r\right\} S, \\[2ex]
\dfrac{d\sigma}{dt} = -\dfrac{1}{na}\left\{\dfrac{2r}{a} - \dfrac{1-e^2}{e}\cos v\right\} R - \dfrac{(1-e^2)}{nae}\left\{1 + \dfrac{r}{p}\right\} \sin v\, S
\end{cases}
$$

XXV PROBLEMS

1 Find the components S and R of this chapter in terms of T and N, which were used in Chapter VIII

$$
Ans\quad \begin{cases}
S = \dfrac{(1+e\cos v)}{\sqrt{1+e^2+2e\cos v}} T + \dfrac{e\sin v}{\sqrt{1+e^2+2e\cos v}} N, \\[2ex]
R = \dfrac{e\sin v}{\sqrt{1+e^2+2e\cos v}} T - \dfrac{1+e\cos v}{\sqrt{1+e^2+2e\cos v}} N
\end{cases}
$$

* If $nt+\sigma$ is replaced by $\int n dt + \sigma'$ as explained in the last of Art 180, a is not to be considered as entering implicitly through n It is under this supposition that this equation is derived

2 By means of the equations of problem 1 express the variations of the elements ☊, , σ in terms of T and N, and verify all the results contained in the table of Art 145

3 Explain why $\frac{d\omega}{dt}$ contains a term depending upon W

4 Suppose the disturbed body moves in a resisting medium, find the equations for the variations of the elements

Ans
$$\left\{\begin{aligned}
\frac{d☊}{dt}&=0,\\
\frac{di}{dt}&=0,\\
\frac{d\omega}{dt}&=\frac{2\sqrt{1-e^2}}{nae}\frac{\sin v}{\sqrt{1+e^2+2e\cos v}}T,\\
\frac{da}{dt}&=\frac{2\sqrt{1+e^2+2e\cos v}}{n\sqrt{1-e^2}}T,\\
\frac{de}{dt}&=\frac{2\sqrt{1-e^2}(\cos v+e)}{na\sqrt{1+e^2+2e\cos v}}T,\\
\frac{d\sigma}{dt}&=-\frac{2(1-e^2)(1+e^2+e\cos v)\sin v}{nae(1+e\cos v)\sqrt{1+e^2+2e\cos v}}T
\end{aligned}\right.$$

5 Discuss the way in which the elements vary in the last problem, including for what values of v the maxima and minima in their rates of change occur, when T is a constant, and when it varies as the square of the velocity

6 Derive the equations corresponding to (77) for the elements ☊, i, π, a, e, and ϵ

Ans
$$\left\{\begin{aligned}
\frac{d☊}{dt}&=\frac{r\sin u}{na^2\sqrt{1-e^2}\sin i}W,\\
\frac{di}{dt}&=-\frac{r\cos u}{na^2\sqrt{1-e^2}}W,\\
\frac{d\pi}{dt}&=2\sin^2\frac{i}{2}\frac{d☊}{dt}+\frac{\sqrt{1-e^2}}{nae}\left\{-R\cos v+S\left(1+\frac{r}{p}\right)\sin v\right\},\\
\frac{da}{dt}&=\frac{2}{n\sqrt{1-e^2}}\left(Re\sin v+S\frac{p}{r}\right),\\
\frac{de}{dt}&=\frac{\sqrt{1-e^2}}{na}\left\{R\sin v+\left(\frac{e+\cos v}{1+e\cos v}+\cos v\right)S\right\},\\
\frac{d\epsilon}{dt}&=-\frac{2rR}{na^2}+\frac{e^2}{1+\sqrt{1-e^2}}\frac{d\pi}{dt}+2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d☊}{dt}
\end{aligned}\right.$$

183 Computation of Perturbations by Mechanical Quadratures The perturbations of the first order are computed under the supposition that the disturbing forces and their effects are the same as they would be if both of the bodies moved continually in the undisturbed orbits Therefore the disturbing forces, and in particular the rectangular components R, S, and W, are functions of the positions in the undisturbed orbits, and since the coordinates are expressible as functions of the time, R, S, and W may be considered as being functions of the time If s represents any element, equations (77) may be written in the form

$$\frac{ds}{dt}=f_s(t),$$

and the perturbations of the elements during the interval $t_n - t_0$ are given by the definite integral*

$$-s^{(0)}+s=\int_{t_0}^{t_n} f_s(t)\,dt, \tag{78}$$

where $s^{(0)}$ is the value of the element s at $t=t_0$

The difficulty of computing this integral arises from the fact that $f_s(t)$ is a very complicated function *Mechanical quadrature* is a means of finding an approximate value of the integral without developing $f_s(t)$ explicitly and finding its primitive

The integral (78) may be interpreted as the area contained between the curve $w=f_s(t)$, the t-axis, and the lines $t=t_0$ and $t=t_n$ Suppose

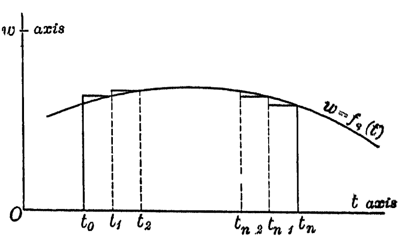

Fig 51

the interval $t_n - t_0$ is divided into n equal spaces and the value of the ordinates at the ends of these spaces computed They are $f_s(t_1)$, $f_s(t_2)$, $\ldots$, $f_s(t_n)$ Then the approximate value of the area in question is

$$s=\sum_{i=1}^{n} f_s(t_i)\,(t_i-t_{i-1}) \tag{79}$$

* See Art 169

This will be more nearly accurate the straighter the curve $w = f_s(t)$ is If it should have rapid changes of direction the error might be relatively great However, in the case under consideration the $f_s(t)$ depends linearly upon the disturbing force, which changes in magnitude slowly as the planets move in their orbits

A method which in general gives better results than the preceding is to pass an algebraic curve of the nth degree through the n points of the curve $w = f_s(t)$ for which $t = t_1, \quad , t_n$, and to find the area contained between this curve, the t-axis, and the same ordinates as before This presents no difficulty because the integral can at once be found for algebraic curves

In general, the degree of accuracy attained by the method of mechanical quadratures depends upon the manner in which the interval $t_n - t_0$ is divided The problem of dividing it in such a manner that the best result shall be obtained, in general, for a given number of divisions was solved by Gauss (*Coll Works*, vol III, *Methodus nova integralium valores per approximationem inveniendi*) If the integral is transformed so that the limits are -1 and $+1$ by the substitution $t' = \frac{-(t_n + t_0) + 2t}{t_n - t_0}$, the points where the divisions may be made most advantageously are, as Gauss proved, given by the equation

$$\frac{d^n (t'^2 - 1)^n}{dt'^n} = 0, \tag{80}$$

all of whose roots are real, distinct, and comprised between -1 and $+1$

If the interval $t_n - t_0$ is divided into enough subintervals the value of the integral will be found within the limits of observation It has been found convenient to use $t_i - t_{i-1} = 40$ days except in the cases of comets passing very near to planets If $f_s(t)$ is of the same sign throughout the interval $t_n - t_0$ the perturbations will be greater the greater t_n is It was shown in Art 171 that the perturbations of the second order are proportional to the terms of the first order, to attain the desired degree of accuracy $t_n - t_0$ must not be taken large enough so that the terms of the second order become sensible For the next interval beginning with t_n the components R, S, and W must be computed with new elements equal respectively to the original elements plus the perturbations in the interval $t_n - t_0$ In this way the process may be continued It will not fail until the work has been carried so far that the accumulated effects of the errors inherent in the

method become sensible, this will in general happen in the course of time, though they be individually inappreciable

In employing mechanical quadratures it is not necessary to express the perturbing forces explicitly in terms of the elements and the time This is of great importance, for, in cases where the eccentricities and inclinations are large, as in some of the asteroid orbits, these expressions, which are series, are very slowly convergent, and in the case of orbits whose eccentricities exceed 0 6627, or of orbits which have any radius of one equal to any radius of the other the series are divergent and cannot be used* The method of mechanical quadratures, on the other hand, is equally applicable to all kinds of orbits, the only restriction being that the intervals shall be taken sufficiently short It is the method actually employed, in one of its many forms, in computing the perturbations of the orbits of comets In addition, it is not necessary to compute terms of the second order, which can be found only by a great deal of labor

The disadvantages are that, in order to find by this method the values of the elements at any particular time, it is necessary to compute them at all of the intermediate epochs Being purely numerical, it throws no light whatever on the general character of perturbations, and leads to no general theorems regarding the stability of a system These are questions of great interest and some of the most brilliant discoveries in Celestial Mechanics have been made respecting them Another objection to the method of mechanical quadratures is that it is mathematically inexact except at the limit as $t_n - t_0$ approaches zero, when it becomes the method of direct integration, by the definition of a definite integral

184 Development of the Perturbative Function In order to apply equations (72) the perturbative function R_1 must be developed explicitly in terms of the elements and the time From this point on perturbations of only the first order will be considered, therefore, in accordance with the results of Art 170, the elements which appear in R_1 are the osculating elements at the time t_0

In the notation of Art 166, the perturbative function is

$$(81)\quad \begin{cases} R_1 = k^2 \left\{ \dfrac{1}{r_{12}} - \dfrac{x_1 x_2 + y_1 y_2 + z_1 z_2}{r_2^3} \right\}, \\ r_1 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}, \\ r_2 = \sqrt{x_2^2 + y_2^2 + z_2^2} \end{cases}$$

* See Arts 99 and 185 (*a*)

The perturbing forces evidently depend upon the mutual inclinations of the orbits, rather than upon their inclinations independently to the fixed plane of reference. It will be convenient, therefore, to develop $R_{1,2}$ in terms of the mutual inclination. Since this angle is expressible in terms of i_1, i_2, Ω_1, and Ω_2 the partial derivatives of $R_{1,2}$ with respect to these elements are to be formed through it, besides so far as they may occur explicitly.

The development of the perturbative function consists of three steps:*

(*a*) Development of $R_{1,2}$ as a power series in the square of the sine of half the mutual inclination of the orbits.

(*b*) Development of the coefficients of the series obtained in (*a*) into power series in e_1 and e_2.

(*c*) Development of the coefficients of the preceding series into Fourier series in the mean longitudes of the two planets and the angular variables π_1, π_2, Ω_1, and Ω_2.

In the little space available here it will not be possible to give more than a general outline of the operations which are necessary to effect the complete development. A detailed discussion is given in Tisserand's *Mécanique Céleste*, vol. I, chapters XII to XVIII inclusive.

185. (*a*) Development in the Mutual Inclination. Let S represent the angle between the radii r_1 and r_2; then

$$\frac{1}{r_{1,2}} = (r_1^2 + r_2^2 - 2r_1 r_2 \cos S)^{-\frac{1}{2}}. \tag{82}$$

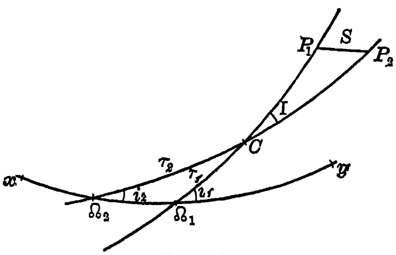

Fig. 52

Let the angles between r_1 and the x, y, and z-axes be α_1, β_1, γ_1 respectively, and in the case of r_2, α_2, β_2, and γ_2. Then it follows that

$$x_1 = r_1 \cos \alpha_1, \quad y_1 = r_1 \cos \beta_1, \quad z_1 = r_1 \cos \gamma_1, \text{ etc.},$$

* There are many more or less important variations of the method outlined here, which is based on the work of Leverrier in the *Annales de l'Observatoire de Paris*, vol. I.

and

$$(83)\quad \begin{cases} x_1x_2+y_1y_2+z_1z_2=r_1r_2(\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2) \\ \qquad\qquad\qquad\quad =r_1r_2\cos S \end{cases}$$

Let I represent the angle between the two orbits, and τ_1 and τ_2 the distances from their ascending nodes to their point of intersection From the spherical triangle P_1P_2C the value of $\cos S$ is found to be

$$(84)\quad \begin{cases} \cos S=\cos(u_1-\tau_1)\cos(u_2-\tau_2)+\sin(u_1-\tau_1)\sin(u_2-\tau_2)\cos I, \text{ or} \\ \cos S=\cos(u_1-u_2+\tau_2-\tau_1)-2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\dfrac{I}{2}, \\ u_1-\tau_1=v_1+\pi_1-\Omega_1-\tau_1, \\ u_2-\tau_2=v_2+\pi_2-\Omega_2-\tau_2 \end{cases}$$

The quantities I, τ_1, and τ_2 are determined by the formulas of Gauss applied to the triangle $\Omega_2\Omega_1C$

$$(85)\quad \begin{cases} \sin I\sin\tau_1=\sin i_2\sin(\Omega_1-\Omega_2), \\ \sin I\sin\tau_2=\sin i_1\sin(\Omega_1-\Omega_2), \\ \sin I\cos\tau_1=\sin i_1\cos i_2-\cos i_1\sin i_2\cos(\Omega_1-\Omega_2), \\ \sin I\cos\tau_2=-\cos i_1\sin i_2+\sin i_1\cos i_2\cos(\Omega_1-\Omega_2), \\ \qquad\cos I=\cos i_1\cos i_2+\sin i_1\sin i_2\cos(\Omega_1-\Omega_2) \end{cases}$$

For simplicity I, τ_1, and τ_2 will be retained, but it must be remembered when the partial derivatives of $R_{1,2}$ are taken that they are functions of i_1, i_2, Ω_1, and Ω_2

As a consequence of (82), (83), and (84) the perturbative function may be written

$$(86)\quad \begin{cases} R_{1,2}=[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{1}{2}} \\ \qquad\qquad\qquad\qquad \left[1+\dfrac{4r_1r_2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\dfrac{I}{2}}{r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)}\right]^{-\frac{1}{2}} \\ -\dfrac{r_1}{r_2^2}\left[\cos(u_1-u_2+\tau_2-\tau_1)-2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\dfrac{I}{2}\right] \end{cases}$$

The radii r_1 and r_2 are independent of I The second factor of the first term of the right member of this equation may be expanded by the binomial theorem into an absolutely converging power series in $\sin^2\dfrac{I}{2}$ so long as

$$\frac{4r_1r_2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\dfrac{I}{2}}{r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)}$$

is less than unity in absolute value This fraction is less than, or at most equal to,

$$\frac{4r_1r_2\sin^2\frac{I}{2}}{(r_2-r_1)^2}$$

If this expression is less than unity for all the values which r_1 and r_2 can take in the given ellipses the expansion of (86) is valid for all values of the time In the case of the major planets it is always very small, the greatest value of $\sin^2\frac{I}{2}$ being for Mercury and Mars, 0 0118 In the perturbations of the planetoids by Jupiter it often fails, for I is sometimes of considerable magnitude while r_2-r_1 may become very small In the case of Mars and Eros r_2-r_1 may actually vanish and this mode of development consequently fails It is needless to say that it is not applicable in the cometary orbits, and on this account mechanical quadratures are employed in these cases

In those cases in which the expansion does not fail

$$(87)\quad \left\{\begin{aligned} R_{1\,2} &= [r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{1}{2}} \\ &\quad - r_1r_2[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{3}{2}} \\ &\qquad\qquad \times 2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\frac{I}{2} \\ &\quad + r_1^2r_2^2[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{5}{2}} \\ &\qquad\qquad \times 6\sin^2(u_1-\tau_1)\sin^2(u_2-\tau_2)\sin^4\frac{I}{2} \\ &\quad + \\ &\quad - \frac{r_1}{r_2^2}\cos(u_1-u_2+\tau_2-\tau_1)+\frac{2r_1}{r_2^2}\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\frac{I}{2} \end{aligned}\right.$$

186 (*b*) **Development in e_1 and e_2** The radii r_1 and r_2 vary from $a_1(1-e_1)$ and $a_2(1-e_2)$ to $a_1(1+e_1)$ and $a_2(1+e_2)$ respectively Let

$$(88)\quad \left\{\begin{aligned} r_1 &= a_1(1+\rho_1), \\ r_2 &= a_2(1+\rho_2) \end{aligned}\right.$$

The angles u_1 and u_2 are expressed in terms of the true anomalies, v_1 and v_2, and the elements by (84) The true anomalies are equal to the mean anomalies plus the equations of the center which may be denoted by w_1 and w_2 Let l_1 and l_2 represent the mean longitudes counted from x [Fig (52)], then

$$(89)\quad \left\{\begin{aligned} u_1-\tau_1 &= l_1-\Omega_1-\tau_1+w_1, \\ u_2-\tau_2 &= l_2-\Omega_2-\tau_2+w_2 \end{aligned}\right.$$

Now R_1 may be written

$$R_{1\,2} = F[a_1(1+\rho_1),\ a_2(1+\rho_2)],$$

where F is a homogeneous function of a_1 and a of degree -1 Therefore

$$(90) \qquad R_{1\,2} = \frac{1}{1+\rho_2} F\left(a_1 + a_1 \frac{\rho_1 - \rho}{1+\rho},\ a_2\right)$$

This equation may be developed by Taylor's formula, giving

$$(91) \quad R_{1\,2} = \frac{1}{1+\rho}\left\{F(a_1,\ a\ ) + \frac{\rho_1 - \rho_2}{1+\rho_2}\frac{a_1}{1}\frac{\partial F(a_1,\ a\ )}{\partial a_1} + \left(\frac{\rho_1 - \rho}{1+\rho}\right)^2 \frac{a_1}{1\ 2}\frac{\partial F(a_1,\ a_2)}{\partial a_1} + \quad \right\}$$

The expressions $\left(\frac{\rho_1 - \rho_2}{1+\rho_2}\right)^i$ may be developed as power series in ρ_1 and ρ_2 But in Art 99, equation (52), ρ is given as a power series in e whose coefficients are cosines of multiples of the mean anomaly Making these expansions and substitutions in (91) $R_{1\,2}$ may be arranged as a power series in e_1 and e_2 These operations are to be actually performed upon the separate terms of the series (87), so the resulting series is arranged according to powers of e_1, e , and $\sin^2\frac{I}{2}$ The angles w_1 and w_2 depend upon e_1 and e_2, but their developments will not be introduced until after the next step

187 (*c*) **Developments in Fourier Series** The first term within the bracket of (91) is obtained by replacing r_1 and r_2 by a_1 and a respectively in (87) The higher terms involve the derivatives of the first with respect to a_1 Referring to the explicit development (87), it is seen that the development of the expressions of the type

$$(a_1 a\ )^{\frac{\nu-1}{2}}\ [a_1^2 + a^2 - 2a_1 a_2 \cos(u_1 - u_2 + \tau_2 - \tau_1)]^{-\frac{\nu}{2}},$$

where ν is an odd integer, must be considered

Let $u_1 - u_2 + \tau_2 - \tau_1 = \psi$ It is known from the theory of Fourier Series that $[a_1 + a_2 - 2a_1 a_2 \cos\psi]^{-\frac{\nu}{2}}$ may be developed into a series of cosines of multiples of ψ, which is convergent for all values of ψ* Thus

$$(92) \qquad (a_1 a\ )^{\frac{\nu-1}{2}}\ [a_1 + a_2^2 - 2a_1 a\ \cos\psi]^{-\frac{\nu}{2}} = \tfrac{1}{2}\sum_{i=-\infty}^{+\infty} B_\nu^{(i)} \cos i\psi,$$

where $B_\nu^{(i)} = B_\nu^{(-i)}$

* See Jordan's *Cours d Analyse*, vol II , chap IV

To find the coefficients $B_\nu^{(i)}$ let $z = e^{\sqrt{-1}\psi}$, where e represents the Napierian base Then

$$2\cos\psi = z + z^{-1}, \quad 2\cos i\psi = z^i + z^{-i}$$

Suppose $a_2 > a_1$ and let $\dfrac{a_1}{a_2} = \alpha$, then (92) becomes

$$\frac{\alpha^{\frac{\nu-1}{2}}}{a_2}(1+\alpha^2-2\alpha\cos\psi)^{-\frac{\nu}{2}} = \tfrac{1}{2}\sum_{-\infty}^{+\infty} B_\nu^{(i)}\cos i\psi \tag{93}$$

Let

$$(1+\alpha^2-2\alpha\cos\psi)^{-\frac{\nu}{2}} = (1-\alpha z)^{-\frac{\nu}{2}}(1-\alpha z^{-1})^{-\frac{\nu}{2}} = \tfrac{1}{2}\sum_{i=-\infty}^{+\infty} b_\nu^{(i)} z^i,$$

whence

$$B_\nu^{(i)} = \frac{\alpha^{\frac{\nu-1}{2}}}{a}\, b_\nu^{(i)} \tag{94}$$

Since the absolute values of αz and αz^{-1} are always less than unity, the factors $(1-\alpha z)^{-\frac{\nu}{2}}$ and $(1-\alpha z^{-1})^{-\frac{\nu}{2}}$ may be expanded by the binomial theorem into convergent power series in αz and αz^{-1} The coefficient of z^i in the product of these series is $\frac{1}{2}b_\nu^{(i)}$, after which $B_\nu^{(i)}$ is obtained from (94) The general term is easily found to be

$$\begin{aligned}\tfrac{1}{2}b_\nu^{(i)} = \frac{\frac{\nu}{2}\left(\frac{\nu}{2}+1\right)\quad\left(\frac{\nu}{2}+i-1\right)}{i\,!}\alpha^i\Bigg[1 &+ \frac{\frac{\nu}{2}}{1}\;\frac{\frac{\nu}{2}+i}{i+1}\alpha^2 \\ &+ \frac{\frac{\nu}{2}\left(\frac{\nu}{2}+1\right)}{1\quad 2}\;\frac{\left(\frac{\nu}{2}+i\right)\left(\frac{\nu}{2}+i+1\right)}{(i+1)(i+2)}\alpha^4 + \quad\Bigg]\end{aligned} \tag{95}$$

In this manner the coefficients of $\rho_1^{j_1}\rho_2^{j_2}\left(\sin^2\dfrac{I}{2}\right)^k$ are developed in Fourier series in $\cos i(u_1-u_2+\tau_2-\tau_1)$ But these trigonometrical functions are multiplied by the factors $\sin(u_1-\tau_1)$, $\sin(u_2-\tau_2)$ raised to different powers [equation (87)] These powers are to be reduced to sines and cosines of multiples of the arguments, and the products formed with $\cos i(u_1-u_2+\tau_2-\tau_1)$ and the reduction again made to sines and cosines of multiples of arcs The trigonometrical terms will have the form $\cos(j_1u_1+j_2u_2+k_1\tau_1+k_2\tau_2)$ where j_1, j_2, k_1 and k_2 are integers As a consequence of (89) this expression may be written

$$\left\{\begin{aligned}&\cos(j_1l_1+j_2l_2-j_1\Omega_1-j_2\Omega_2+k_1\tau_1+k_2\tau_2+j_1w_1+j_2w_2)\\ &=\cos(j_1l_1+j_2l_2-j_1\Omega_1-j_2\Omega_2+k_1\tau_1+k_2\tau_2)\\ &\qquad\qquad\times\{\cos(j_1w_1)\cos(j_2w_2)-\sin(j_1w_1)\sin(j_2w_2)\}\\ &-\sin(j_1l_1+j_2l_2-j_1\Omega_1-j_2\Omega_2+k_1\tau_1+k_2\tau_2)\\ &\qquad\qquad\times\{\sin(j_1w_1)\cos(j_2w_2)+\cos(j_1w_1)\sin(j_2w_2)\}\end{aligned}\right. \tag{96}$$

Since $l_1 = n_1 t + \epsilon_1$, $l_2 = n_2 t + \epsilon$ the first factors of the terms in the right member of this equation are independent of e_1 and e Cos $(j_1 w_1)$, etc are to be expanded into power series in w_1 and w by the usual methods Now $w_1 = v_1 - M_1$, $w = v_2 - M$, and these quantities were developed into power series in e_1 and e_2 [Art 99, eq (54)] whose coefficients were Fourier series with multiples of the mean anomaly as arguments Substituting these series for w_1 and w in the expansions of the second factors of the terms of the right member of (96), and reducing the powers of sines and cosines of the mean anomaly to sines and cosines of multiples of the mean anomaly, and multiplying by the factors

$$\cos(j_1 l_1 + j_2 l_2 - j_1 \Omega_1 - j_2 \Omega + k_1 \tau_1 + k \tau)$$

and

$$\sin(j_1 l_1 + j_2 l_2 - j_1 \Omega_1 - j_2 \Omega_2 + k_1 \tau_1 + k_2 \tau_2)$$

and again reducing to sines and cosines of multiples of the arguments, the expression (96) is developed as a power series in e_1 and e_2 whose coefficients are series in sines and cosines of sums of multiples of $l_1, l_2, \Omega_1, \Omega_2, \tau_1, \tau_2, M_1, M_2$ But $M_1 = l_1 - \pi_1$, $M = l_2 - \pi$, therefore the arguments will be $l_1, l, \Omega_1, \Omega, \tau_1, \tau, \pi_1, \pi_2$, where τ_1 and τ_2 are functions of Ω_1, Ω, i_1, and i defined by (85)

When these expansions and reductions have all been made $R_{1\,2}$ will be developed in a power series in e_1, e_2, and $\sin^2 \frac{I}{2}$, the coefficients of which are series of sines and cosines of multiples of $l_1, l, \Omega_1, \Omega_2, \tau_1, \tau, \pi_1, \pi_2$, the coefficient of each trigonometric term depending upon the ratio of the major semi-axes A more detailed discussion than can be entered into here shows many important and interesting properties of these expansions, one of which is that the development of $R_{1\,2}$ contains cosine terms alone Thus

$$(97)\quad \begin{cases} R_{1\,2} = \Sigma C \cos D, \\ D = j_1(n_1 t + \epsilon_1) + j(n_2 t + \epsilon_2) - j_1' \Omega_1 - j_2' \Omega_2 \\ \qquad\qquad + k_1 \tau_1 + k_2 \tau_2 + k_1' \pi_1 + k' \pi_2, \\ C = f\left(a_1, a_2, e_1, e, \sin^2 \frac{I}{2}\right), \end{cases}$$

in which $j_1, \ldots, k_2'$ take all integral values, positive, negative, and zero, the summation being extended over all of these terms

The derivatives with respect to the various elements may at once be found, and as the time enters only in the trigonometrical terms there is no trouble in integrating term by term It is clear from the foregoing that the series for $R_{1\,2}$ is very complicated and that much labor is required in constructing it in a particular case Leverrier has

carried out the literal development of all terms up to the seventh order inclusive in e_1, e_2, $\sin^2\frac{I}{2}$, and the length of the work is such that fifty-three quarto pages of the first volume of the *Annales de l'Observatoire de Paris* are required in order to write out the result

188 Periodic Variations It follows from equations (72) and (97) that the rates of change of the elements are given by

$$
(98)\quad \left\{
\begin{aligned}
\frac{d\Omega_1}{dt} &= \frac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\,\Sigma\left\{\frac{\partial C}{\partial i_1}\cos D-\left(k_1\frac{\partial \tau_1}{\partial i_1}+k_2\frac{\partial \tau_2}{\partial i_1}\right)C\sin D\right\},\\
\frac{di_1}{dt} &= \frac{-m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\,\Sigma\left\{j_1'-k_1\frac{\partial \tau_1}{\partial \Omega_1}-k_2\frac{\partial \tau_2}{\partial \Omega_1}\right\}C\sin D\\
&\quad+\frac{m_2\tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\,\Sigma\left\{k_1+j_1+k_1\frac{\partial \tau_1}{\partial \pi_1}+k_2\frac{\partial \tau_2}{\partial \pi_1}\right\}C\sin D,\\
\frac{d\pi_1}{dt} &= \frac{m_2\tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\,\Sigma\left\{\frac{\partial C}{\partial i_1}\cos D-\left(k_1\frac{\partial \tau_1}{\partial i_1}+k_2\frac{\partial \tau_2}{\partial i_1}\right)C\sin D\right\}\\
&\quad+\frac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\,\Sigma\frac{\partial C}{\partial e_1}\cos D,\\
\frac{da_1}{dt} &= \frac{-2m_2}{n_1 a_1}\,\Sigma j_1 C\sin D,\\
\frac{de_1}{dt} &= m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\,\Sigma j_1 C\sin D\\
&\quad+\frac{m_2\sqrt{1-e_1^2}}{n_1 a_1\, e_1}\,\Sigma\left\{k_1'+k_1\frac{\partial \tau_1}{\partial \pi_1}+k_2\frac{\partial \tau_2}{\partial \pi_1}\right\}C\sin D,\\
\frac{d\epsilon_1}{dt} &= \frac{m_2\tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\,\Sigma\left\{\frac{\partial C}{\partial i_1}\cos D-\left(k_1\frac{\partial \tau_1}{\partial i_1}+k_2\frac{\partial \tau_2}{\partial i_1}\right)C\sin D\right\}\\
&\quad+m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\,\Sigma\frac{\partial C}{\partial e_1}\cos D-\frac{2m_2}{n_1 a_1}\,\Sigma\frac{\partial C}{\partial a_1}\cos D
\end{aligned}
\right.
$$

The perturbations of the elements of m_1 of the first order with respect to the mass m_2 are the integrals of these equations regarding the elements as constants in the right members Similar terms must be added for each disturbing planet

There are terms in $R_{1,2}$ of three classes (*a*) those in which $j_1 n_1 + j_2 n_2$ is finite and distinct from zero, (*b*) those in which $j_1 n_1 + j_2 n_2$ is very small, but distinct from zero, and (*c*) those in which $j_1 n_1 + j_2 n_2$ equals

zero Denote the fact that $R_{1,2}$ contains these three sorts of terms by writing

$$R_{1,2}=\Sigma C_0\cos D_0+\Sigma C_1\cos D_1+\Sigma C\ \cos D\ ,$$

where the three sums in the right member include these three classes of terms respectively Hence the perturbations of the first order and of the first kind are, including the mass factor m,

$$
(99)\left\{
\begin{aligned}
(\Omega_1^{(0,1)})&=\frac{m_2}{n_1a_1^2\sqrt{1-e_1^2}\sin i_1}\Sigma\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1n_1+j_2n_2}\right.\\
&\qquad\left.+\left(k_1\frac{\partial\tau_1}{\partial i_1}+k_2\frac{\partial\tau_2}{\partial i_1}\right)\frac{C_0\cos D_0}{j_1n_1+j_2n_2}\right\}-(\Omega_1^{(0,1)})_{t_0},\\
(i_1^{(0,1)})&=\frac{m}{n_1a_1^2\sqrt{1-e_1^2}\sin i_1}\Sigma\left\{j_1'-k_1\frac{\partial\tau_1}{\partial\Omega_1}-k_2\frac{\partial\tau_2}{\partial\Omega_1}\right\}\frac{C_0\cos D_0}{j_1n_1+j_2n_2}\\
&\quad-\frac{m_2\tan\frac{i_1}{2}}{n_1a_1\sqrt{1-e_1^2}}\Sigma\left\{k_1'+j_1+k_1\frac{\partial\tau_1}{\partial\pi_1}+k_2\frac{\partial\tau_2}{\partial\pi_1}\right\}\frac{C_0\cos D_0}{j_1n_1+j_2n_2}-(i_1^{(0,1)})_{t_0},\\
(\pi_1^{(0,1)})&=\frac{m_2\tan\frac{i_1}{2}}{n_1a_1^2\sqrt{1-e_1^2}}\Sigma\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1n_1+j_2n_2}\right.\\
&\qquad\left.+\left(k_1\frac{\partial\tau_1}{\partial i_1}+k_2\frac{\partial\tau_2}{\partial i_1}\right)\frac{C_0\cos D_0}{j_1n_1+j_2n_2}\right\}\\
&\qquad+\frac{m_2\sqrt{1-e_1^2}}{n_1a_1e_1}\Sigma\frac{\partial C_0}{\partial e_1}\frac{\sin D_0}{j_1n_1+j_2n_2}-(\pi_1^{(0,1)})_{t_0},\\
(a_1^{(0,1)})&=\frac{2m_2}{n_1a_1}\Sigma j_1\frac{C_0\cos D_0}{j_1n_1+j_2n_2}-(a_1^{(0,1)})_{t_0},\\
(e_1^{(0,1)})&=-m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1a_1^2e_1}\Sigma j_1\frac{C_0\cos D_0}{j_1n_1+j_2n_2}\\
&\quad-\frac{m_2\sqrt{1-e_1^2}}{n_1a_1^2e_1}\Sigma\left(k_1'+k_1\frac{\partial\tau_1}{\partial\pi_1}+k_2\frac{\partial\tau_2}{\partial\pi_1}\right)\frac{C_0\cos D_0}{j_1n_1+j_2n_2}-(e_1^{(0,1)})_{t_0},\\
(\epsilon_1^{(0,1)})&=\frac{m_2\tan\frac{i_1}{2}}{n_1a_1^2\sqrt{1-e_1^2}}\Sigma\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1n_1+j_2n_2}\right.\\
&\quad\left.+\left(k_1\frac{\partial\tau_1}{\partial i_1}+k_2\frac{\partial\tau_2}{\partial i_1}\right)\frac{C_0\cos D_0}{j_1n_1+j_2n_2}\right\}+m_2\sqrt{1-e_1^2}\frac{1-\sqrt{1-e_1^2}}{n_1a_1^2e_1}\Sigma\frac{\partial C_0}{\partial e_1}\frac{\sin D_0}{j_1n_1+j_2n_2}\\
&\qquad-\frac{2m_2}{n_1a_1}\Sigma\frac{\partial C_0}{\partial a_1}\frac{\sin D_0}{j_1n_1+j_2n_2}-(\epsilon_1^{(0,1)})_{t_0}
\end{aligned}
\right.
$$

These terms are purely periodic with periods $\dfrac{2\pi}{j_1n_1+j_2n_2}$, and constitute the *periodic variations* Every element is subject to them,

depending upon an infinity of such terms, whose periods are, in general, different The coefficient C_0 is, in general, smaller the larger $j_1 n_1 + j_2 n_2$ is, and the shorter is the period of the term

This method of representing the motion of the planets is somewhat analogous to the epicycloid theory of Ptolemy, for each term alone is equivalent to the adding of a small circular motion to that previously existing This theory is more complex than that of Ptolemy in that it adds cycloid upon cycloid without limit, it is simpler than that of Ptolemy in that it flows from one simple law, the law of gravitation

189 Long Period Variations The letters j_1 and j_2 represent all positive and negative integers and zero Therefore, unless n_1 and n_2 are incommensurable such a pair of integers j_1 and j_2 exists that $j_1 n_1 + j_2 n_2 = 0$ But in this case D would be a constant and the integral would not be formed this way However, whether n_1 and n_2 are incommensurable or not such a pair of numbers can be found that $j_1 n_1 + j_2 n_2$ is very small This term will be large unless C is very small It is shown in a complete discussion of the development of $R_{1\,2}$ that the order of C in e_1, e_2, $\sin^2 \dfrac{I}{2}$ is at the least equal to the numerical value of $j_1 + j_2$, (see Tisserand's *Méc Cél*, vol I p 308) Since n_1 and n_2 are both positive one of the numbers j_1, j must be positive and the other negative in order to make the sum $j_1 n_1 + j_2 n_2$ small, and the more nearly equal they are the smaller the numerical value of $j_1 + j_2$ is, and consequently, the larger C will be When the mean motions of the two planets are such that they are nearly commensurable with the ratio of n_1 to n_2 expressible in small numbers, then large terms in the perturbations will arise from the presence of these small divisors The period of such a term is $\dfrac{2\pi}{j_1 n_1 + j_2 n_2}$, which is very long, whence the appellation *long period* These terms are given by equations of the same form as (99), but with the restriction that $j_1 n_1 + j_2 n_2$ shall be very small

Geometrically considered, the condition that the periods shall be nearly commensurable with the ratio expressible in small numbers means that the points of conjunction occur at nearly the same part of the orbits with only a few other conjunctions intervening The extreme case is that in which there are no conjunctions intervening, when j_1 and j_2 will differ in numerical value by unity

The mean motions of Jupiter and Saturn are nearly in the ratio of

five to two Consequently $j_1 = 2$, $j_2 = -5$ gives a long period term, and the order of the coefficient C is the absolute value of $2 - 5$, or 3 The cause of the long period inequality of Jupiter and Saturn was discovered by Laplace in 1784 in computing the perturbations of the third order in e_1 and e_2 The length of the period in the case of these two planets is about 850 years

190 Secular Variations The expression D is independent of the time for all of those terms in which $j_1 = j_2 = 0$ The partial derivatives of D with respect to the elements are also independent of the time, hence, taking these terms of (98) and integrating, it is found that

$$
(100)\quad
\begin{cases}
\begin{aligned}
[\Omega_1^{(0,1)}] &= \frac{m_2}{n_1 a_1 \sqrt{1-e_1^2}\sin i_1} \\
&\qquad \times \Sigma\left\{\frac{\partial C}{\partial i_1}\cos D - \left(k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau}{\partial i_1}\right) C \sin D_2\right\}(t-t_0), \\
[i_1^{(0,1)}] &= \frac{-m}{n_1 a_1 \sqrt{1-e_1}\sin i_1}\,\Sigma\left\{j_1 - k_1\frac{\partial \tau_1}{\partial \Omega_1} - k_2\frac{\partial \tau}{\partial \Omega_1}\right\} C_2 \sin D\ (t-t_0) \\
&\quad + \frac{m \tan\frac{i_1}{2}}{n_1 a_1^2 \sqrt{1-e_1}}\,\Sigma\left\{k_1' + k_1\frac{\partial \tau_1}{\partial \pi_1} + k\frac{\partial \tau_2}{\partial \pi_1}\right\} C_2 \sin D\ (t-t_0), \\
[\pi_1^{(0,1)}] &= \frac{m \tan\frac{i_1}{2}}{n_1 a_1 \sqrt{1-e_1^2}}\,\Sigma\left\{\frac{\partial C}{\partial i_1}\cos D_2 - \left(k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau}{\partial i_1}\right) C_2 \sin D_2\right\}(t-t_0) \\
&\quad + \frac{m_2\sqrt{1-e_1}}{n_1 a_1 e_1}\,\Sigma\frac{\partial C}{\partial e_1}\cos D_2\ (t-t_0), \\
[a_1^{(0,1)}] &= 0, \\
[e_1^{(0,1)}] &= \frac{m_2\sqrt{1-e_1}}{n_1 a_1^2 e_1}\,\Sigma\left\{k_1' + k_1\frac{\partial \tau_1}{\partial \pi_1} + k_2\frac{\partial \tau_2}{\partial \pi_1}\right\} C_2 \sin D_2\ (t-t_0), \\
[\epsilon_1^{(0,1)}] &= \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2 \sqrt{1-e_1}}\,\Sigma\left\{\frac{\partial C_2}{\partial i_1}\cos D - \left(k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right) C_2 \sin D_2\right\}(t-t_0) \\
&\quad + m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\,\Sigma\frac{\partial C_2}{\partial e_1}\cos D_2\ (t-t_0) \\
&\quad - \frac{2m_2}{n_1 a_1}\,\Sigma\frac{\partial C_2}{\partial a_1}\cos D_2\ (t-t_0)
\end{aligned}
\end{cases}
$$

It follows that there are no secular terms of this type of the first order with respect to the masses in the perturbations of a

This constitutes the first theorem on the stability of the solar system It was proved up to the second powers of the eccentricities by Laplace in 1773*, when he was but twenty-four years of age, in a memoir upon the mutual perturbations of Jupiter and Saturn, it was shown by Lagrange in 1776 that it is true for all powers of the eccentricities† It was proved by Poisson in 1809 that there are no secular terms in a in the perturbations of the second order with respect to the masses, but that there are terms of the type $t \cos D$, where D contains the time‡ Terms of this type are commonly called Poisson terms

All of the elements except a have secular terms The thought appears to have been that the secular terms, which apparently cause the elements to change without limit, alone prevent the use of equations (72) for computing the perturbations for any time however great Many methods of computing perturbations have been devised in order to avoid the appearance of secular terms, yet it is clear that, whether terms proportional to the time appear or not, the method is strictly valid for only those values of the time for which the series (20) of Art 168 are convergent

Secular terms may enter in another way, usually not considered If $j_1 n_1 + j_2 n_2 = 0$ with $j_1 \neq 0$, $j_2 \neq 0$, D is independent of the time and the corresponding terms are secular In this case D is not independent of ϵ_1 and there will be secular terms in the perturbations of a As has been remarked, this condition will always be fulfilled by an infinity of values of j_1 and j_2 if n_1 and n_2 are not incommensurable But it is impossible to determine from observations whether n_1 and n_2 are incommensurable, for there is always a limit to the accuracy with which observations can be made, and within this limit there always exist both commensurable and incommensurable numbers There is as much reason, therefore, to say that secular terms in a of this type exist as that they do not However, they are of no practical importance because the ratio of n_1 to n_2 cannot be expressed in small integers, and the coefficients of these terms are so small that they are not sensible for such values of the time as are ordinarily used

191 Terms of the Second Order with Respect to the Masses The terms of the second order are defined by equations (29)

* Memoir presented to the *Paris Academy of Sciences*

† *Memoirs of the Berlin Academy*, 1776

‡ *Journal de l'Ecole Polytechnique*, vol xv

Art 171 The right members of these equations are the products of the partial derivatives with respect to the elements, of the right members which occur in the terms of the first order, and the perturbations of the first order of the corresponding elements Thus, the second order perturbations of the node are determined by the equations

$$
(101)\qquad
\begin{cases}
\dfrac{d\Omega_1^{(0,2)}}{dt} = \dfrac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\,\sin i_1}\displaystyle\sum_{s_1}\frac{\partial^2 R_1}{\partial i_1 \partial s_1}\, s_1^{(0,1)}, \\[2ex]
\dfrac{d\Omega_1^{(1,1)}}{dt} = \dfrac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\,\sin i_1}\displaystyle\sum_{s_2}\frac{\partial^2 R_1}{\partial i_1 \partial s_2}\, s_2^{(1,0)},
\end{cases}
$$

where s_1 and s_2 represent the elements of the orbits of m_1 and m_2 respectively $\dfrac{\partial^2 R_1}{\partial i_1 \partial s_1}$ is a sum of periodic and constant terms, $s_1^{(0,1)}$ and $s_2^{(1,0)}$ are sums of periodic terms and terms containing the time to the first degree as a factor The products $\dfrac{\partial^2 R_1}{\partial i_1 \partial s_1}\, s_1^{(0,1)}$ and $\dfrac{\partial^2 R_1}{\partial i_1 \partial s_2}\, s_2^{(1,0)}$ will contain terms of four types (a), $\frac{\sin}{\cos} D$, where D contains the time, (b), $t \frac{\sin}{\cos} D$, (c), $\frac{\sin}{\cos} D$, where D is independent of the time, and (d), $t \frac{\sin}{\cos} D$ The integrals of these four types are respectively

$$
(a),\ \frac{\frac{-\cos}{\sin} D}{j_1 n_1 + j_2 n_2}, \qquad (b),\ t\,\frac{\frac{-\cos}{\sin} D}{j_1 n_1 + j_2 n_2} + \frac{\frac{\sin}{\cos} D}{(j_1 n_1 + j_2 n_2)^2},
$$

$$
(c),\ t \frac{\sin}{\cos} D, \quad \text{and} \quad (d),\ \frac{t^2}{2} \frac{\sin}{\cos} D.
$$

Therefore, the perturbations of the second order with respect to the masses have purely periodic terms, Poisson terms, or terms in which the trigonometric terms are multiplied by the time, secular terms where the time occurs to the first degree, and secular terms where the time occurs to the second degree This is true for all of the elements except the major semi-axis, in the case of which the coefficients of the terms of the third and fourth types are zero, as Poisson first proved

In the terms of the third order with respect to the masses there are secular terms in the perturbations of all the elements except a_1, which are proportional to the third power of the time, and so on

192 Lagrange's Treatment of the Secular Variations The presence of the secular terms in the expressions for the elements seems to indicate that, if it be assumed that the series represent the elements for all values of the time, then the elements change without limit with the time But this is by no means necessarily so For example, consider the function

$$\sin(cmt) = cmt - \frac{c\,m^3t^3}{3} + \quad , \tag{102}$$

where c is a constant and m a very small factor which may take the place of a mass The expansion converges for all values of t This function is never greater than unity for any value of the time, yet if its expansion in powers of m were given, and if the first few terms were considered without the law of the coefficients being known, it might seem that the series represents a function which increases indefinitely in numerical value with the time

Following out the idea that the secular terms may be expansions of functions which are always finite Lagrange has shown (see *Collected Works*, vols V and VI), under certain assumptions which have not been logically justified, that the secular terms are in reality the expansions of periodic terms of very long period These terms differ from the long period variations (Art 189) in that they come from the small uncompensated parts of the periodic variations, instead of directly from special conditions of conjunctions As a rule these terms are very small, and their periods are much longer than those of the sensible long period terms It will not be possible to give here more than a very general idea of the method of Lagrange

The first step is a transformation of variables by the equations

$$\begin{cases} h_j = e_j \sin \pi_j, \\ l_j = e_j \cos \pi_j, \end{cases} \tag{103}$$

and

$$\begin{cases} p_j = \tan i_j \sin \Omega_j, \\ q_j = \tan i_j \cos \Omega_j, \end{cases} \tag{104}$$

where e_j, π_j, etc are the elements of the orbit of m_j, and l_j is a new variable not to be confused with the mean longitude These transformations are to be made simultaneously in the elements of the orbits of all of the planets The elements a_j and ϵ_j remain without transformation Omitting the subscripts, it is found from (103) and (104) that

$$
(105)\quad \begin{cases}
\dfrac{dh}{dt} = e\cos\pi\dfrac{d\pi}{dt} + \sin\pi\dfrac{de}{dt}, \\[2ex]
\dfrac{de}{dt} = -e\sin\pi\dfrac{d\pi}{dt} + \cos\pi\dfrac{de}{dt}, \\[2ex]
\dfrac{\partial R}{\partial e} = \dfrac{\partial R}{\partial h}\dfrac{\partial h}{\partial e} + \dfrac{\partial R}{\partial l}\dfrac{\partial l}{\partial e} = \sin\pi\dfrac{\partial R}{\partial h} + \cos\pi\dfrac{\partial R}{\partial l}, \\[2ex]
\dfrac{\partial R}{\partial \pi} = \dfrac{\partial R}{\partial h}\dfrac{\partial h}{\partial \pi} + \dfrac{\partial R}{\partial l}\dfrac{\partial l}{\partial \pi} = e\cos\pi\dfrac{\partial R}{\partial h} - e\sin\pi\dfrac{\partial R}{\partial l} \\[2ex]
\dfrac{dp}{dt} = \tan\iota\cos\Omega\dfrac{d\Omega}{dt} + \sec^2\iota\sin\Omega\dfrac{d\iota}{dt}, \\[2ex]
\dfrac{dq}{dt} = -\tan\iota\sin\Omega\dfrac{d\Omega}{dt} + \sec^2\iota\cos\Omega\dfrac{d\iota}{dt}, \\[2ex]
\dfrac{\partial R}{\partial \Omega} = \dfrac{\partial R}{\partial p}\dfrac{\partial p}{\partial \Omega} + \dfrac{\partial R}{\partial q}\dfrac{\partial q}{\partial \Omega} = \tan\iota\cos\Omega\dfrac{\partial R}{\partial p} - \tan\iota\sin\Omega\dfrac{\partial R}{\partial q}, \\[2ex]
\dfrac{\partial R}{\partial \iota} = \dfrac{\partial R}{\partial p}\dfrac{\partial p}{\partial \iota} + \dfrac{\partial R}{\partial q}\dfrac{\partial q}{\partial \iota} = \sec^2\iota\sin\Omega\dfrac{\partial R}{\partial p} + \sec^2\iota\cos\Omega\dfrac{\partial R}{\partial q}
\end{cases}
$$

Then it follows from (72) that

$$
(106)\quad \begin{cases}
\dfrac{dh}{dt} = \dfrac{m\sqrt{1-h-l}}{na}\dfrac{\partial R}{\partial l} \\[1ex]
\qquad - \dfrac{m\sqrt{1-h^2-l}}{na^2}\,\dfrac{h}{1+\sqrt{1-h-l^2}}\,\dfrac{\partial R}{\partial \epsilon} + \dfrac{m\, l\tan\frac{\iota}{2}}{na\sqrt{1-h^2-l}}\,\dfrac{\partial R}{\partial \iota}, \\[2ex]
\dfrac{dl}{dt} = \dfrac{-m_2\sqrt{1-h^2-l}}{na^2}\dfrac{\partial R}{\partial h} \\[1ex]
\qquad - \dfrac{m_2\sqrt{1-h^2-l^2}}{na^2}\,\dfrac{l}{1+\sqrt{1-h^2-l^2}}\,\dfrac{\partial R}{\partial \epsilon} - \dfrac{m\, h\tan\frac{\iota}{2}}{na^2\sqrt{1-h^2-l^2}}\,\dfrac{\partial R}{\partial \iota}, \\[2ex]
\dfrac{dp}{dt} = \dfrac{m_2}{na^2\sqrt{1-h^2-l^2}\cos^3\iota}\dfrac{\partial R}{\partial q} - \dfrac{m_2 p}{2na^2\sqrt{1-h^2-l}\cos\iota\cos^2\frac{\iota}{2}}\left(\dfrac{\partial R}{\partial \pi} + \dfrac{\partial R}{\partial \epsilon}\right), \\[2ex]
\dfrac{dq}{dt} = \dfrac{-m_2}{na^2\sqrt{1-h^2-l^2}\cos^3\iota}\dfrac{\partial R}{\partial p} - \dfrac{m_2 q}{2na^2\sqrt{1-h^2-l^2}\cos\iota\cos^2\frac{\iota}{2}}\left(\dfrac{\partial R}{\partial \pi} + \dfrac{\partial R}{\partial \epsilon}\right)
\end{cases}
$$

Developing the right members and neglecting all terms of degree higher than the first* in h, l, p, and q, these equations reduce to

* The terms of order higher than the first are neglected throughout in a later step in the method

(107)
$$\begin{cases} \dfrac{dh}{dt} = \dfrac{m_2}{na^2}\dfrac{\partial R}{\partial l}, \\ \dfrac{dl}{dt} = -\dfrac{m_2}{na^2}\dfrac{\partial R}{\partial h}, \\ \dfrac{dp}{dt} = \dfrac{m_2}{na^2}\dfrac{\partial R}{\partial q}, \\ \dfrac{dq}{dt} = -\dfrac{m_2}{na^2}\dfrac{\partial R}{\partial p} \end{cases}$$

Each perturbing planet will contribute terms in the right members of these equations similar to the ones written which come from m_2 These differential equations are not strictly correct, since the first approximation has already been made in neglecting the higher powers of the variables

The second step is the new method of treating the differential equations The expansions of the R_j contain certain terms which are independent of the time, which in the ordinary method give rise to the secular terms Let $R_j^{(0)}$ represent these terms Lagrange then treated the differential equations by neglecting the periodic terms in R_j, and writing

(108)
$$\begin{cases} \dfrac{dh_i^{(0)}}{dt} = \sum_j m_j \dfrac{\partial R_{i,j}^{(0)}}{\partial l_i}, \\ \dfrac{dl_i^{(0)}}{dt} = -\sum m_j \dfrac{\partial R_{i,j}^{(0)}}{\partial h_i}, \\ \dfrac{dp_i^{(0)}}{dt} = \sum m_j \dfrac{\partial R_{i,j}^{(0)}}{\partial q_i}, \\ \dfrac{dq_i^{(0)}}{dt} = -\sum m_j \dfrac{\partial R_{i,j}^{(0)}}{\partial p_i} \end{cases}$$

The values of h, l, p, and q determined from these equations were used instead of the secular terms obtained by the previous method There is no known mathematical justification for breaking up a differential equation in this manner, and it is at this point that the processes lack perfect rigor, even though it is agreed to neglect the terms of higher orders

The right members of equations (108) are expanded in powers of h, l, p, and q, and all of the terms except those of the first degree are neglected, consequently the terms omitted in (107) would have disappeared here if they had been retained up to this point The system

becomes linear, and the detailed discussion of $R_{i,j}$ shows that it is homogeneous, giving equations of the form

$$(109)\qquad \begin{cases} \dfrac{dh_1}{dt} - \sum\limits_{j=1}^{n} c_{1j} l_j = 0, \\ \dfrac{dl_1}{dt} + \sum\limits_{j=1}^{n} c_{1j} h_j = 0, \\ \dfrac{dh_2}{dt} - \sum\limits_{j=1}^{n} c_{2j} l_j = 0, \\ \dfrac{dl_2}{dt} + \sum\limits_{j=1}^{n} c_{2j} h_j = 0, \\ \qquad \cdots \\ \dfrac{dh_n}{dt} - \sum\limits_{j=1}^{n} c_{nj} l_j = 0, \\ \dfrac{dl_n}{dt} + \sum\limits_{j=1}^{n} c_{nj} h_j = 0, \end{cases}$$

and a similar system of equations in p_j and q_j

The coefficients c_{ij} depend only on the major axes (the ϵ_j not appearing in the secular terms) which are considered as being constants, since the major axes have no secular terms in the perturbations of the first and second orders with respect to the masses It is to be noted here that there is no justification of the assumption that the c_{ij} are constants

When these linear equations are solved by the method used in Art 123, the values of the variables are found in the form

$$(110)\qquad \begin{cases} h_i = \sum\limits_{j=1}^{n} H_{ij} e^{\lambda_j t}, \\ l_i = \sum\limits_{j=1}^{n} L_{ij} e^{\lambda_j t}, \\ p_i = \sum\limits_{j=1}^{n} P_{ij} e^{\mu_j t}, \\ q_i = \sum\limits_{j=1}^{n} Q_{ij} e^{\mu_j t}, \end{cases}$$

where H_{ij}, L_{ij}, P_{ij}, Q_{ij}, are constants depending upon the initial conditions A detailed discussion shows that the λ_j and μ_j are all pure imaginaries with very small absolute values, therefore the h_i, l_i, p_i, and q_i oscillate around mean values with very long periods Or, since the e_j and $\tan i_j$ are expressible as the sums of squares of the

h_j, l_j, p_j, and q_j, it follows that they also perform small oscillations with long periods, for example, the eccentricity of the earth's orbit is now decreasing and will continue to decrease for about 24,000 years

Equations (109) admit integrals first found by Laplace in 1784, which lead practically to the same theorem They are

$$(111)\qquad \begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 (h_j^2 + l_j^2) = \text{Constant} = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 (p_j^2 + q_j^2) = C', \end{cases}$$

or, because of (102) and (103),

$$(112)\qquad \begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 e_j^2 = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 \tan^2 i_j = C', \end{cases}$$

where n_j is the mean motion of m_j The constants C and C' as determined by the initial conditions are very small, and since the left members of (112) are made up of positive terms alone no e_j or i_j can ever become very great There might be an exception if the corresponding m_j were very small compared to the others

These equations give the celebrated theorems of Laplace that the eccentricities and inclinations cannot vary except within very narrow limits Although the demonstration lacks complete rigor, yet the results must be considered as remarkable and significant Equations (112) do not give the periods and amplitudes of the oscillations as do equations (110)

XXVI PROBLEMS

1 Suppose (*a*) that $R_{1\,2}$ is large and nearly constant, (*b*) that $R_{1\,2}$ is large and changing rapidly, (*c*) that $R_{1\,2}$ is small and nearly constant If the perturbations are computed by mechanical quadratures how should the $t_n - t_0$ be chosen relatively in the three cases, and how should the numbers of subdivisions of $t_n - t_0$ compare?

2 The perturbative function involves the reciprocal of the distance from the disturbing to the disturbed planets This is called the *principal part* and gives the most difficulty in the development How many separate reciprocal distances must be developed in order to compute, in a system of one sun and

n planets, (a) the perturbations of the first order of one planet, (b) the perturbations of the first order of two planets, (c) the perturbations of the second order of one planet, and (d) the perturbations of the third order of one planet?

3. What simplifications would there be in the development of the perturbative function if the mutual inclinations of the orbits were zero, and if the orbits were circles?

4. What sorts of terms will in general appear in perturbations of the third order with respect to the masses?

HISTORICAL SKETCH AND BIBLIOGRAPHY

The theory of perturbations, as applied to the Lunar Theory, was developed from the geometrical standpoint by Newton. The memoirs of Clairaut and D'Alembert in 1747 contained important advances, making the solutions depend upon the integration of the differential equations in series. Clairaut soon had occasion to apply his processes of integration to the perturbations of Halley's comet by the planets Jupiter and Saturn. This comet had been observed in 1531, 1607, and 1682. If its period were constant it would pass the perihelion again about the middle of 1859. Clairaut computed the perturbations due to the attractions of Jupiter and Saturn, and predicted that the perihelion passage would be April 13, 1859. He remarked that the time was uncertain to the extent of a month because of the uncertainties in the masses of Jupiter and Saturn and the possibility of perturbations from unknown planets beyond these. The comet passed the perihelion March 13, giving a striking proof of the value of Clairaut's methods.

The theory of the perturbations of the planets was begun by Euler, whose memoirs on the mutual perturbations of Jupiter and Saturn gained the prizes of the French Academy in 1748 and 1752. In these memoirs was given the first analytical development of the method of the variation of parameters. The equations were not entirely general as he had not considered the elements as being all simultaneously variables. The first steps in the development of the perturbative function were also given by Euler.

Lagrange, whose contributions to Celestial Mechanics were of the most brilliant character, wrote his first memoir in 1766 on the perturbations of Jupiter and Saturn. In this work he developed still further the method of the variation of parameters, leaving his final equations, however, still incorrect by regarding the major axes and the epochs of the perihelion passages as constants in deriving the equations for the variations. The equations for the inclination, node, and longitude of the perihelion from the node were

perfectly correct In the expressions for the mean longitudes of the planets there were terms proportional to the first and second powers of the time These were entirely due to the imperfections of the method, their true form being that of the long period terms, as was shown by Laplace in 1784 by considering terms of the third order in the eccentricities The method of the variation of parameters was completely developed for the first time in 1782 by Lagrange in a prize memoir on the perturbations of comets moving in elliptical orbits

In 1773 Laplace presented his first memoir to the French Academy of Sciences In it he proved his celebrated theorem that, up to the second powers of the eccentricities, the major axes, and consequently the mean motions, have no secular terms This theorem was extended by Lagrange in 1774 and 1776 to all powers of the eccentricities and of the sine of the angle of the mutual inclination, for perturbations of the first order with respect to the masses Poisson proved in 1809 that the major axes have no purely secular terms in the perturbations of the second order with respect to the masses Finally, Haretu proved in his Dissertation at Sorbonne in 1878 that there *are* secular variations in the expressions for the major axes in the terms of the third order with respect to the masses

Lagrange began the study of the secular terms in 1774, introducing the variables h, l, p, and q The investigations were carried on by Lagrange and Laplace, each supplementing and extending the work of the other, until 1784 when their work became complete by Laplace's discovery of his celebrated equations

$$\begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 e_j^2 = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 \tan^2 i_j = C' \end{cases}$$

These equations were derived by using only the linear terms in the differential equations Leverrier, Hill, and others have extended the work by methods of successive approximations to terms of higher degree Newcomb (*Smithsonian Contributions to Science*, vol XXI, 1876) has established the more far reaching results that it is possible, in the case of the planetary perturbations, to represent the elements by purely periodic functions of the time which formally satisfy the differential equations of motion If these series were convergent the stability of the solar system would be assured, but Poincaré has shown that they are in general divergent (*Les Méthodes Nouvelles*, chap IX) Lindstedt and Gyldèn have also succeeded in integrating the equations of the motion of n bodies in periodic series, which, however, are in general divergent

Gauss, Airy, Adams, Leverrier, Hansen, and many others have made important contributions to the planetary theory in some of its many aspects Adams and Leverrier are noteworthy for having predicted the existence and apparent position of Neptune from the unexplained irregularities in the motion of Uranus More recently Poincaré has turned his attention to

Celestial Mechanics, publishing a prize memoir in the *Acta Mathematica*, vol XIII This has been enlarged and published in book form with the title *Les Methodes Nouvelles de la Mécanique Celeste* Poincaré has applied to the problem all the resources of modern mathematics with unrivalled genius, he has brought into this work such a wealth of ideas and he has devised methods of such immense power that the subject in its theoretical aspects has been entirely revolutionized in his hands It cannot be doubted that much of the work of the next fifty years will be in amplifying and applying the processes which he has explained

The following works should be consulted

Laplace's *Mecanique Céleste*, containing practically all that was known of Celestial Mechanics at the time it was written (1799—1805)

On the variation of parameters—*Annales de l'Observatoire de Paris*, vol I, Tisserand's *Mecanique Celeste*, vol I, Brown's *Lunar Theory*, Dziobek's *Planeten Bewegungen*

On the development of the perturbative function—*Annales de l'Observatoire de Paris*, vol I, Tisserand's *Mécanique Celeste*, vol I, Hansen's *Entwickelung des Products einer Potenz des Radius Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie*, etc, *Abh d K Sächs Ges zu Leipzig*, vol II

Newcomb's memoir on the General Integrals of Planetary Motion

Poincaré, *Les Méthodes Nouvelles*, vol I chap VI

On the stability of the solar system—Tisserand's *Méc Cél* vol I chaps XI, XXV, XXVI, and vol IV chap XXVI, Gyldèn, *Traité Analytique des Orbites absolues*, vol I, Newcomb, *Smithsonian Cont*, vol XXI, Poincaré, *Les Méthodes Nouvelles*, vol II chap X

On the subject of Celestial Mechanics as a whole there is no better work available than that of Tisserand, which should be in the possession of every one giving special attention to this subject

CHAPTER X

THEORY OF THE DETERMINATION OF THE ELEMENTS OF PARABOLIC ORBITS

Preparation of the Observations

193 The determination of the orbits of unknown bodies is based upon observations of their apparent directions from the observer on the surface of the earth at given epochs It is obvious that their actual directions must be known in order that the elements may be computed, and it is convenient to have the actual directions from the center of the earth rather than from any point on its surface, because the center is the point which describes the curve known as the earth's orbit The position of the sun with respect to the center of the earth is given in the *Nautical Almanac* for every day in the year Since the comets are the bodies to which this theory will be applied it will be assumed at once that the body in question is a comet

The chief corrections to be applied to the observations before using them in the computation of an orbit are for parallax, the time it takes light to come from the comet to the earth, aberration, and the changes in the fundamental circles of reference during the interval covered by the observations These corrections are all relatively small, especially when the comet is not near the earth, and if they were neglected an approximate orbit would be found Moreover, it is necessary to know the approximate distance of the comet in order to apply some of them

Suppose a comet has just been discovered An orbit should be computed as soon as the requisite number of observations has been made, and an ephemeris calculated, in order that observers may be directed where to point their telescopes If this were not done, and observations were prevented for a few days by unfavorable weather, the comet might move so that its rediscovery would require the

expenditure of considerable time But an approximate orbit will be sufficient for the construction of a search ephemeris unless the comet should remain visible for a long time, therefore it is recommended that the first orbit be computed without applying the corrections which are about to be explained After the comet has receded from view and all the observations have been made a definitive orbit should be computed based upon all of the available data to which all of the corrections have been applied

194 Correction for Parallax Suppose the approximate distances of the comet at the times of the observations are known, then the corrections for parallax can be made by the methods of Spherical and Practical Astronomy It will be sufficient to give the formulas with the proper references

Let h represent the radius of the earth expressed in terms of the equatorial radius, π the equatorial horizontal parallax of the sun expressed in seconds of arc, $\theta - \alpha$ the hour angle of the comet at the time of the observation, ρ the distance of the comet, ϕ the apparent astronomical latitude of the observer, ϕ' the geocentric latitude, and e the eccentricity of the meridian Then the geocentric latitude is given by the equation

$$\tan\phi' = (1-e)\tan\phi\text{*}, \tag{1}$$

where ϕ' is to be taken in the same quadrant as ϕ

If α_0 and δ_0 represent the observed right ascension and declination, the values of these two coordinates corrected for parallax are given by†

$$(2)\qquad \begin{cases} \alpha = \alpha_0 + \dfrac{h\pi\cos\phi'\sin(\theta-\alpha_0)}{\rho\cos\delta_0}, \\ \tan\gamma = \dfrac{\tan\phi'}{\cos(\theta-\alpha_0)}, \quad 0<\gamma<180, \\ \delta = \delta_0 + \dfrac{h\pi\sin\phi'\sin(\gamma-\delta_0)}{\rho\sin\gamma}, \end{cases}$$

in which the corrections to α_0 and δ_0 are expressed in seconds of arc

195 The Locus Fictus Gauss has devised a method which avoids the necessity of correcting for parallax‡ It consists in treating the observation as though it were made from the point where the line from the comet through the true place of the observer pierces the plane

* Chauvenet *Sph and Prac Ast*, vol I p 98

† Chauvenet *Sph and Prac Ast*, vol I p 125 Campbell's *Sph and Prac Ast*, p 32

‡ *Theoria Motus*, Art 72

of the ecliptic This point is called the *locus fictus* Since a complete observation determines the direction of this line it is obvious that it determines the locus fictus uniquely, and when the observation is supposed to have been made from this point the latitude of the sun is always zero

Let t' represent the sidereal time of the observation, and ϕ' the geocentric latitude of the observer Then t' and ϕ' are the right ascension and declination of the geocentric zenith Let λ' and β' represent the celestial longitude and latitude of the geocentric zenith, they are found from t' and ϕ' by equations (71), Chap V Then, with the definitions of h and π of the last article, the geocentric coordinates x', y', z' of the place of observation with the center of the earth as origin are given by

$$\begin{cases} x' = h\pi \sin 1'' \cos\beta' \cos\lambda', \\ y' = h\pi \sin 1'' \cos\beta' \sin\lambda', \\ z' = h\pi \sin 1'' \sin\beta' \end{cases}$$

Let ρ and ρ' represent the distance from the comet to the place of observation and to the locus fictus respectively, and λ and β the longitude and latitude of the comet, which are to be computed from the observed right ascension and declination by equations (71), Chap V Then the coordinates of the place of observation referred to the locus fictus are

$$\begin{cases} x'' = (\rho' - \rho) \cos\beta \cos\lambda, \\ y'' = (\rho' - \rho) \cos\beta \sin\lambda, \\ z'' = (\rho' - \rho) \sin\beta \end{cases}$$

Let the coordinates of the sun referred to the center of the earth as origin be X, Y, Z, and to the locus fictus, X', Y', 0 Then

$$\begin{aligned} X' &= X - x' + x'', \\ Y' &= Y - y' + y'', \\ 0 &= Z - z' + z'', \end{aligned}$$

or, in polar coordinates,

$$(3)\quad \begin{cases} R' \cos\Lambda' = R\cos\Lambda\cos\mathrm{B} - h\pi\sin 1'' \cos\beta' \cos\lambda' + (\rho' - \rho)\cos\beta\cos\lambda, \\ R' \sin\Lambda' = R\sin\Lambda\cos\mathrm{B} - h\pi\sin 1'' \cos\beta' \sin\lambda' + (\rho' - \rho)\cos\beta\sin\lambda, \\ \qquad 0 = R\sin\mathrm{B} \quad - h\pi\sin 1'' \sin\beta' \quad + (\rho' - \rho)\sin\beta \end{cases}$$

These equations define R' and Λ', which are to be used throughout the computation in place of R and Λ, if the locus fictus is employed When the observed latitude is small, the distance from the center of the earth to the locus fictus is very great and the method cannot be employed with advantage

196 Reduction of the Time Because of the finite velocity with which light travels the apparent direction of the comet at any instant is the direction which it had at some previous time, viz, that at which the observed rays left the comet Either the apparent position of the comet, or the time of the observation may be corrected The latter is evidently the simpler, since it depends upon the distance of the comet alone and does not necessitate the recomputation of the auxiliaries which depend upon the spherical coordinates Light travels a unit's distance in 498 65 seconds, therefore the correction to be subtracted from the time of observation is 498 65 ρ, or, if the observations be referred to the locus fictus, 498 65 ρ'

The geocentric distance, ρ, to be used is that at the corrected time which is unknown But, as the geocentric distance in general changes very slowly, the correction to the time taking the ρ at the uncorrected time will be sensibly correct If in any case it should not a second correction may be computed using the ρ at a more nearly correct time

197 Correction for Aberration The observations will be affected by the aberrations which arise from the motion of the earth around the sun, and from its rotation on its axis Let α_0 and δ_0 represent the observed right ascension and declination of the comet, and α and δ the same coordinates when corrected for the annual aberration They are given by the equations*

$$(4)\quad \begin{cases} \alpha = \alpha_0 + 20''\ 481 \sec \delta_0 (\cos \Lambda \cos \alpha_0 \cos \epsilon + \sin \Lambda \sin \alpha_0), \\ \delta = \delta_0 + 20''\ 481 \cos \Lambda (\cos \delta_0 \sin \epsilon - \sin \alpha_0 \sin \delta_0 \cos \epsilon) \\ \qquad + 20''\ 481 \sin \Lambda \cos \alpha_0 \sin \delta_0 \end{cases}$$

The corrections arising from the diurnal aberration, which must be subtracted from the observed coordinates, are †

$$(5)\quad \begin{cases} 0''\ 322 \cos \phi \cos (\theta - \alpha_0) \sec \delta_0 & \text{(in right ascension)}, \\ 0''\ 322 \cos \phi \sin (\theta - \alpha_0) \sin \delta_0 & \text{(in declination)}, \end{cases}$$

where ϕ is the latitude of the observer, and $\theta - \alpha_0$ the hour angle of the comet at the time of observation

198 Reduction to the Mean Equinox The coordinates will be affected by precession and nutation It is customary to refer

* Chauvenet, *Sph and Prac Ast*, vol I p 633 (The numerical coefficient based on older observations is 20 445 in Chauvenet instead of 20 481 ± 0 008, as given by Nyrén)

† Chauvenet, *Sph and Prac Ast*, vol I p 640 (Chauvenet gives for the numerical coefficient 0 311 instead of 0 322, which is found by more recent determinations)

them to the mean equinox at the beginning of the year. As before, let α_0 and δ_0 represent the observed coordinates. Then the coordinates referred to the mean equinox at the beginning of the year are given by*

$$(6)\qquad \begin{cases} \alpha = \alpha_0 - 15f - g \sin (G + \alpha_0) \tan \delta_0, \\ \delta = \delta_0 - g \cos (G + \alpha_0), \end{cases}$$

where f, g, and G are the *independent star-numbers*, which are given in the *Nautical Almanac* for every day in the year. The corrections are expressed in seconds of arc.

If the aberrations have not been computed by the methods of the last article, they may be included in one set of equations with the precession and nutation, when the formulas become†

$$(7)\qquad \begin{cases} \alpha = \alpha_0 - 15f - g \sin (G + \alpha_0) \tan \delta_0 - h \sin (H + \alpha_0) \sec \delta_0, \\ \delta = \delta_0 - g \cos (G + \alpha_0) - h \cos (H + \alpha_0) \sin \delta_0 - i \cos \delta_0. \end{cases}$$

The quantities h, H, and i are also given in the *Nautical Almanac*.

The various corrections explained above are so small that it is immaterial in what order they are applied.

General Considerations

199. Formulation of Problem. The constants of integration in Chap. V were determined in terms of the initial conditions, that is, in terms of the initial coordinates and components of velocity. When a heavenly body is discovered, as for example a comet, not all of the initial conditions are determined by observation, since the direction from the earth is the only thing that is observed. It is clear then that the elements of the orbit cannot be determined unless additional observations are made.

In one complete observation two things are given, the two angular coordinates of the body, which determine its direction, while the distance and components of velocity are unknown. If the equatorial system is the one used, α and δ are observed, and ρ is unknown. In order to determine the six elements of the orbit six things must be observed; or, three complete observations are required. Suppose the observations are made at the instants t_1, t_2, and t_3. Let the corresponding coordinates have the subscripts 1, 2, and 3 respectively.

* Chauvenet, *Sph. and Prac. Ast.*, vol. I, Arts. 402—404.

† See the American *Ephemeris* and *Nautical Almanac* for 1902, p. 290.

The coordinates are functions of the elements, which may be indicated by writing

$$
(8)\qquad \begin{cases}
\alpha_1 = \phi\,(\Omega,\ i,\ \pi,\ a,\ e,\ T,\ t_1), \\
\alpha\ \ = \phi\,(\qquad\qquad\qquad\quad,\ t_2), \\
\alpha_3 = \phi\,(\qquad\qquad\qquad\quad,\ t_3), \\
\delta_1 = \psi\,(\qquad\qquad\qquad\quad,\ t_1), \\
\delta_2 = \psi\,(\qquad\qquad\qquad\quad,\ t_2), \\
\delta_3 = \psi\,(\qquad\qquad\qquad\quad,\ t_3)
\end{cases}
$$

Thus there are six equations involving six unknowns The solution for the elements Ω, , T constitutes the solution of the problem The functions ϕ and ψ are transcendental, involving the elements in a very complicated fashion, as was shown in Chap V, while the coordinates were found by passing through Kepler's Equation, or the cubic in the case of the parabola, and by means of a number of trigonometrical transformations Therefore equations (8) cannot be solved directly by elementary processes

200 Intermediate Elements Although the ultimate object is to determine the elements, the problem of finding certain other quantities from which the elements can be found may be treated first As has been shown, if the coordinates and components of velocity are known at any time the elements can be determined Suppose it is desired to find the coordinates and components of velocity at t_1, then the equations corresponding to (8) become

$$
(9)\qquad \begin{cases}
\alpha_1 = \alpha_1, \\
\alpha\ \ = f(\alpha_1,\ \delta_1,\ \rho_1,\ x_1',\ y_1,\ z_1',\ t_2), \\
\alpha_3 = f(\alpha_1,\ \delta_1,\ \rho_1,\ x_1,\ y_1',\ z_1',\ t_3), \\
\delta_1 = \delta_1, \\
\delta_{\_} = g\,(\alpha_1,\ \delta_1,\ \rho_1,\ x_1',\ y_1',\ z_1',\ t\ ), \\
\delta_3 = g\,(\alpha_1,\ \delta_1,\ \rho_1,\ x_1',\ y_1',\ z_1',\ t_3),
\end{cases}
$$

where $x_1' = \dfrac{dx_1}{dt}$, etc

α_1 and δ_1 are observed quantities and the first and fourth equations may be suppressed The problem is therefore reduced to the solution of only four simultaneous equations, and, moreover, they are somewhat simpler than equations (8)

As another set of intermediate elements the coordinates at two epochs may be taken If they are given and the interval of time which it takes the body to move from one position to the other, the

elements can be determined as will be shown in the proper place It will be sufficient to show here by the most elementary considerations that it is true for parabolic orbits, to which the method will be first applied

The two positions and the center of the sun determine the plane of the orbit Then the focus of the parabola and two of its points are given Construct circles around the given points with radii equal to the distances to the focus One of the two tangents to these two circles will be the directrix of the parabola, the one to be taken being determined by the time it has taken the comet to move from one point to the other The solution will be ambiguous only when the time given by the law of areas is the same in both parabolas This will seldom, if ever, happen, especially since the observations upon which the work is based are generally near together But if any ambiguity should remain, it may be removed by means of the second observation

Suppose the coordinates which are to be found are those of the comet at t_1 and t_3 They are all observed except ρ_1 and ρ_3 Then the equations corresponding to (8) and (9) are

$$(10)\qquad \begin{cases} \alpha_2 = \theta(\rho_1, \rho_3, t_2), \\ \delta_2 = \chi(\rho_1, \rho_3, t_2) \end{cases}$$

By this choice of intermediate elements the problem is reduced to the solution of two simultaneous equations, which are, however, very complicated Nevertheless, the problem has been immensely simplified in comparison with that expressed by equations (8) Instead of using (10) any two independent functions of these equations may be used, or, any function of them with an independent equation involving ρ_1 and ρ_3 Such a function is evidently Euler's equation (see Art 92) When the function of (10) is properly taken the solution is by the famous method of Dr Olbers which was published in the *Berliner Jahrbuch* for 1833, and which will be made the basis of the work on parabolic orbits which is to follow Owing to the transcendental nature of the coefficients in one of the equations involved, methods of approximation will be necessarily employed

201 General Algebraic Solution When the orbit is assumed to be parabolic a rigorous algebraic solution is possible The coördinates of the body at two epochs will be used as the intermediate elements The unknowns in the positions are the geocentric distances ρ_1, ρ_2, and ρ_3

It will be convenient in what follows to use rectangular coordinates, and it must be remembered that they are algebraic functions of the geocentric distances of the first degree, since

$$x_1 = \rho_1 \cos \alpha_1 \cos \delta_1 - X_1, \text{ etc}$$

The motion of the comet is in a plane passing through the center of the sun, therefore the coordinates must at every instant verify the equation

$$(11) \qquad Ax + By + Cz = 0$$

There are but two independent constants in this equation, $\frac{B}{A}$ and $\frac{C}{A}$ if $A \neq 0$, and they are for the present unknown Therefore there are the following conditions upon the coordinates and the unknown constants

$$(12) \qquad \begin{cases} x_1 + \frac{B}{A} y_1 + \frac{C}{A} z_1 = 0, \\ x_2 + \frac{B}{A} y \;\; + \frac{C}{A} z_2 = 0, \\ x_3 + \frac{B}{A} y_3 + \frac{C}{A} z_3 = 0 \end{cases}$$

These equations depend only upon the fact that the motion is in a plane passing through the sun

The comet moves in a parabola lying in the plane $Ax + By + Cz = 0$ with the origin at the focus, or, it lies on the surface of a paraboloid in space with the origin at the focus The general equation of a paraboloid is of the second degree in x, y, and z The question is how many new unknown constants are involved in the equation of the one upon which the body moves If p is the parameter of a parabola its equation with the origin at the focus and x axis as the principal axis is

$$y^2 = 2px + 4p^2$$

The only unknown is p But the principal axis may lie in any direction in space This introduces as new unknowns the two direction cosines, but there is one relation between them since the axis of the parabola must lie in the plane $Ax + By + Cz = 0$ If this parabola be rotated around its axis the paraboloid is generated upon which the body must always lie From what has just been said it follows that the equation of the paraboloid involves only two new unknown constants, which will be represented by p and D Let $F\left(x, y, z, \frac{B}{A}, \frac{C}{A}, p, D\right) = 0$

represent the equation of the paraboloid Then the following conditions must be fulfilled

$$(13) \qquad \begin{cases} F\left(x_1, y_1, z_1, \dfrac{B}{A}, \dfrac{C}{A}, p, D\right) = 0, \\ F\left(x_2, y_2, z_2, \dfrac{B}{A}, \dfrac{C}{A}, p, D\right) = 0, \\ F\left(x_3, y_3, z_3, \dfrac{B}{A}, \dfrac{C}{A}, p, D\right) = 0 \end{cases}$$

The other independent relations among the coordinates are given by Euler's equation, one for the epochs t_1 and t_2, and another for the epochs t_2 and t_3 When the moving body is a comet its mass may be neglected and M will equal unity Then Euler's equation gives

$$(14) \qquad \begin{cases} 6k\,(t_2 - t_1) = (r_1 + r_2 + s_{12})^{\frac{3}{2}} \mp (r_1 + r_2 - s_{12})^{\frac{3}{2}}, \\ 6k\,(t_3 - t_2) = (r_2 + r_3 + s_{23})^{\frac{3}{2}} \mp (r_2 + r_3 - s_{23})^{\frac{3}{2}} \end{cases}$$

The quantities r_1, r_2, r_3, s_{12}, and s_{23} are expressible algebraically in terms of the coordinates, therefore when equations (14) are rationalized they are rational algebraic functions of the coordinates Moreover, they do not introduce any new unknown quantities Therefore equations (12), (13), and (14) are eight independent functions of the coordinates involving algebraically the seven unknowns ρ_1, ρ_2, ρ_3, $\dfrac{B}{A}$, $\dfrac{C}{A}$, p, and D The eight equations are consistent if the motion is in a parabola, and as has been shown, the solution of the problem is in general unique

Suppose that all of the unknowns except one have been eliminated from equations (14) by means of the first six equations This can be done in various ways, as by the methods of Bezout, Euler, and Sylvester* Suppose the greatest common divisor of the last two equations, which contain but one unknown, is found If the solution of the problem is unique it will be of the first degree and will determine the value of the unknown This may be substituted into the next to the last step in the elimination, when a repetition of the process will give the value of another unknown This may be continued until all the unknowns have been found, after which the elements are easily determined. The method is all that could be desired from a theoretical standpoint, being valid for observations distributed in any manner whatever Practically, it would be extremely laborious owing to the very high degrees of the equations involved, and it is unnecessary to say that it has never been applied The above is an explanation of Poincaré's statement, in the

* See Serret's *Algèbre Supérieure*, vol I chap IV

preface to Tisserand's *Leçons sur la Détermination des Orbites*, that the problem of the determination of the elements of a parabolic orbit depends upon the solution of algebraic equations

In the case of elliptic orbits equations (12) are the same, equations (13) involve a new unknown depending upon the eccentricity, and equations (14) are transcendental, being power series in $\frac{1}{a}$ The method fails in this case because the number of unknowns equals the number of equations and because the equations are not all algebraic

Olbers' Method

202 Outline of Olbers' Method Suppose three complete observations at the epochs t_1, t_2, and t_3 are at the disposal of the computer Since the motion is in a plane it follows that

$$
(15) \qquad \left\{ \begin{aligned} Ax_1 + By_1 + Cz_1 &= 0, \\ Ax_2 + By_2 + Cz &= 0, \\ Ax_3 + By_3 + Cz_3 &= 0 \end{aligned} \right.
$$

Eliminating the unknown constants the determinant

$$
\begin{vmatrix} x_1, & y_1, & z_1 \\ x\ , & y\ , & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = 0
$$

is obtained This determinant may be expanded and written in the three forms

$$
(16) \qquad \left\{ \begin{aligned} & x_1(y_2z_3 - z_2y_3) - x\ (y_1z_3 - z_1y_3) + x_3(y_1z_2 - z_1y_2) = 0, \\ & y_1(x_2z_3 - z_2x_3) - y\ (x_1z_3 - z_1x_3) + y_3(x_1z_2 - z_1x\ ) = 0, \\ & z_1(x_2y_3 - y\ x_3) - z_2(x_1y_3 - y_1x_3) + z_3(x_1y_2 - y_1x_2) = 0 \end{aligned} \right.
$$

Evidently these equations are but different forms of the same equation, but, if the parentheses can be determined numerically from some additional principles, they become independent The parentheses are the projections of the triangles formed by the sun and the three positions of the comet taken in twos upon the three fundamental planes Since the parentheses enter so that the factor of projection is the same for each term in the homogeneous equations they may be

replaced by the corresponding triangles Denote the triangle contained between r_i and r_j by $[r_i, r_j]$, then equations (16) may be written

$$(17)\qquad \begin{cases} \dfrac{[r_2, r_3]}{[r_1, r_3]} x_1 + \dfrac{[r_1, r_2]}{[r_1, r_3]} x_3 = x_2, \\[2ex] \dfrac{[r_2, r_3]}{[r_1, r_3]} y_1 + \dfrac{[r_1, r_2]}{[r_1, r_3]} y_3 = y_2, \\[2ex] \dfrac{[r_2, r_3]}{[r_1, r_3]} z_1 + \dfrac{[r_1, r_2]}{[r_1, r_3]} z_3 = z_2 \end{cases}$$

Suppose for the moment that the ratios of the triangles are known, then equations (17) involve only the three unknowns ρ_1, ρ_2, and ρ_3, since $x_1 = \rho_1 \cos\alpha_1 \cos\delta_1 - X_1$ etc Apparently they could be solved for these three quantities, in which they are linear, after which the elements could easily be computed, but, it would be found, as will be shown later, that very small divisors would be introduced so that the solutions would be numerically indeterminate, or at least subject to great uncertainties This is especially so since the ratios of the triangles cannot be found in the first approximation with perfect exactness

In developing formulas for computations it must be always borne in mind that decimals beyond the sixth or seventh place are neglected and that the last one retained is subject to considerable uncertainty owing to the possibility that the neglected parts may accumulate and lead to a considerable error If a quantity depends upon the quotient of two very small numbers the last digits retained have a relatively more important effect on the result, which, therefore, partakes more largely of their uncertainty Consequently, formulas which involve small divisors are to be avoided for practical reasons, even though they may be theoretically correct

Olbers' method consists in eliminating but one unknown, ρ_2, and that from such of the three equations that the single equation which results shall be as simple as possible Since the ρ_i enter linearly the resulting equation will have the form

$$(18)\qquad \rho_3 = m + M\rho_1$$

In addition to equation (18) Euler's equation,

$$(19)\qquad 6k\,(t_3 - t_1) = (r_1 + r_3 + s)^{\frac{3}{2}} \mp (r_1 + r_3 - s)^{\frac{3}{2}},$$

is used This is expressible in terms of ρ_1 and ρ_3 alone as unknowns since r_1, r_3, and s are simple functions of the coordinates, and k and $t_3 - t_1$ are known numbers Consequently equations (18) and (19)

enable one to find the two geocentric distances ρ_1 and ρ_3, from which, together with the other coordinates at these epochs, the elements can be found.

203. Explicit Development of Olbers' Equations. The observations give the apparent position of the comet in right ascension and declination, but it is more convenient in the work which follows to use longitude and latitude. Therefore the coordinates must be transformed by equations (71), Chap. V. The heliocentric rectangular coordinates are expressed in terms of the four angles and the geocentric distances by the equations

$$(20)\qquad \begin{cases} x=\rho\cos\beta\cos\lambda-R\cos\mathrm{B}\cos\Lambda, \\ y=\rho\cos\beta\sin\lambda-R\cos\mathrm{B}\sin\Lambda, \\ z=\rho\sin\beta\qquad\qquad\; -R\sin\mathrm{B}, \end{cases}$$

with the subscripts 1, 2, and 3 for the three observations. Since the latitude of the sun never exceeds 0″.9 B may be put equal to zero in the first approximation, and it always is exactly zero if the locus fictus is employed. Then equations (17) become

$$(21)\qquad \begin{cases} \dfrac{[r_2, r_3]}{[r_1, r_3]}(\rho_1\cos\beta_1\cos\lambda_1-R_1\cos\Lambda_1)+\dfrac{[r_1, r_2]}{[r_1, r_3]}(\rho_3\cos\beta_3\cos\lambda_3-R_3\cos\Lambda_3) \\ \qquad\qquad\qquad\qquad =\rho_2\cos\beta_2\cos\lambda_2-R_2\cos\Lambda_2, \\ \dfrac{[r_2, r_3]}{[r_1, r_3]}(\rho_1\cos\beta_1\sin\lambda_1-R_1\sin\Lambda_1)+\dfrac{[r_1, r_2]}{[r_1, r_3]}(\rho_3\cos\beta_3\sin\lambda_3-R_3\sin\Lambda_3) \\ \qquad\qquad\qquad\qquad =\rho_2\cos\beta_2\sin\lambda_2-R_2\sin\Lambda_2, \\ \dfrac{[r_2, r_3]}{[r_1, r_3]}\rho_1\sin\beta_1+\dfrac{[r_1, r_2]}{[r_1, r_3]}\rho_3\sin\beta_3=\rho_2\sin\beta_2 \end{cases}$$

The unknown ρ_2 will be eliminated from these equations, and evidently it may be done in the three different ways in which the three equations can be combined in twos. The character of the observations determines which one is best adapted, in any particular case, to the solution of the problem.

204. First Method of Eliminating ρ_2. The second and third equations of (21) will be used first. Rotate the axes forward in the plane of the ecliptic through the angle Λ_2, and let

$$(22)\qquad \begin{cases} L_1=R_1\sin(\Lambda_1-\Lambda_2), \\ L_3=R_3\sin(\Lambda_3-\Lambda_2) \end{cases}$$

Then the second equation of (21) becomes

$$\frac{[r_2,\ r_3]}{[r_1,\ r_3]}\{\rho_1 \cos\beta_1 \sin(\lambda_1-\Lambda_2)-L_1\}+\frac{[r_1,\ r_2]}{[r_1,\ r_3]}\{\rho_3 \cos\beta_3 \sin(\lambda_3-\Lambda_2)-L_3\}$$
$$=\rho_2 \cos\beta_2 \sin(\lambda_2-\Lambda_2)$$

Eliminating ρ_2 from this equation by the third equation of (21), it is found that

$$\rho_3 = m' + M'\rho_1, \tag{23}$$

where

$$(24)\quad \begin{cases} m' = \dfrac{1}{\cos\beta_3 \sin(\lambda_3-\Lambda_2)-\sin\beta_3 \cot\beta_2 \sin(\lambda_2-\Lambda_2)}\left\{\dfrac{[r_2,\ r_3]}{[r_1,\ r_2]}L_1+L_3\right\}, \\ M' = \dfrac{[r_2,\ r_3]}{[r_1,\ r_2]}\,\dfrac{\sin\beta_1 \cot\beta_2 \sin(\lambda_2-\Lambda_2)-\cos\beta_1 \sin(\lambda_1-\Lambda_2)}{\cos\beta_3 \sin(\lambda_3-\Lambda_2)-\sin\beta_3 \cot\beta_2 \sin(\lambda_2-\Lambda_2)} \end{cases}$$

The trigonometrical coefficients in these equations may be so transformed that they may be more easily computed, and so that it is easier to see when they become indeterminate

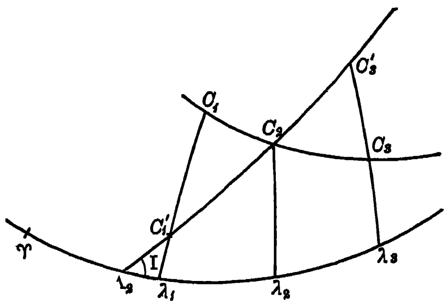

Fig 53

Let $\Upsilon\Lambda_2\lambda_3$ be the ecliptic, and C_1, C_2, and C_3 the apparent positions of the comet at the epochs t_1, t_2, and t_3 respectively Then

$$\lambda_1 C_1 = \beta_1, \qquad \lambda_2 C_2 = \beta_2, \qquad \lambda_3 C_3 = \beta_3$$

Pass a great circle through C_2 and Λ_2, and let

$$\lambda_1 C_1' = \beta_1', \qquad \lambda_3 C_3' = \beta_3', \qquad \text{angle } C_2\Lambda_2\lambda_2 = I$$

From the right triangles it follows that

$$(25)\quad \begin{cases} \sin(\lambda_2-\Lambda_2) = \tan\beta_2 \cot I, \\ \sin(\lambda_1-\Lambda_2) = \tan\beta_1' \cot I, \\ \sin(\lambda_3-\Lambda_2) = \tan\beta_3' \cot I, \end{cases}$$

the first of which defines $\cot I$, and the second and third β_1' and β_3' respectively. Equations (24) become, as a consequence of (25),

$$(26)\qquad \begin{cases} m' = \dfrac{\tan I \cos\beta_3'}{\sin(\beta_3'-\beta_3)}\left\{\dfrac{[r_2, r_3]}{[r_1, r_2]} L_1 + L_3\right\}, \\[2ex] M' = \dfrac{[r_2, r_3]}{[r_1, r_2]}\,\dfrac{\sin(\beta_1-\beta_1')}{\sin(\beta_3'-\beta_3)}\,\dfrac{\cos\beta_3'}{\cos\beta_1'}. \end{cases}$$

205. Second Method of Eliminating ρ_2. The first and third equations of (21) will now be used. Rotating the axis forward in the plane of the ecliptic through the angle Λ_2, and eliminating ρ_2 from the first equation of (21) by means of the third, it follows that

$$(27)\qquad \rho_3 = m'' + M''\rho_1,$$

where

$$\begin{cases} m'' = \dfrac{\sin\beta_2\{[r_2, r_3] R_1\cos(\Lambda_1-\Lambda_2)+[r_1, r_2] R_3\cos(\Lambda_3-\Lambda_2)-[r_1, r_3] R_2\}}{[r_1, r_2]\{\sin\beta_2\cos\beta_3\cos(\lambda_3-\Lambda_2)-\sin\beta_3\cos\beta_2\cos(\lambda_2-\Lambda_2)\}}, \\[2ex] M'' = \dfrac{[r_2, r_3]}{[r_1, r_2]}\,\dfrac{\sin\beta_1\cos\beta_2\cos(\lambda_2-\Lambda_2)-\sin\beta_2\cos\beta_1\cos(\lambda_1-\Lambda_2)}{\sin\beta_2\cos\beta_3\cos(\lambda_3-\Lambda_2)-\sin\beta_3\cos\beta_2\cos(\lambda_2-\Lambda_2)}. \end{cases}$$

These equations may be simplified in a manner analogous to that employed in the last article by introducing convenient auxiliaries. For this purpose let β_1'' and β_3'' be such auxiliaries that

$$\begin{cases} \cos(\lambda_1-\Lambda_2) = \dfrac{\tan\beta_1''}{\tan\beta_2}\cos(\lambda_2-\Lambda_2), \\[2ex] \cos(\lambda_3-\Lambda_2) = \dfrac{\tan\beta_3''}{\tan\beta_2}\cos(\lambda_2-\Lambda_2), \end{cases}$$

then

$$(28)\qquad \begin{cases} m'' = \dfrac{\sin\beta_2\cos\beta_3'\{[r_2, r_3] R_1\cos(\Lambda_1-\Lambda_2)+[r_1, r_2] R_3\cos(\Lambda_3-\Lambda_2)-[r_1, r_3] R_2\}}{[r_1, r_2]\cos\beta_2\cos(\lambda_2-\Lambda_2)\sin(\beta_3''-\beta_3)}, \\[2ex] M'' = \dfrac{[r_2, r_3]}{[r_1, r_2]}\,\dfrac{\sin(\beta_1-\beta_1'')}{\sin(\beta_3''-\beta_3)}\,\dfrac{\cos\beta_3''}{\cos\beta_1''}. \end{cases}$$

206. Third Method of Eliminating ρ_2. The first and second equations of (21) will now be used. Multiply the first equation by $\sin\lambda_2$ and the second by $-\cos\lambda_2$ and add, and it follows that

$$(29)\qquad \rho_3 = m''' + M'''\rho_1,$$

where

$$(30)\qquad \begin{cases} m''' = \dfrac{[r_2, r_3] R_1\sin(\Lambda_1-\lambda_2)+[r_1, r_2] R_3\sin(\Lambda_3-\lambda_2)-[r_1, r_3] R_2\sin(\Lambda_2-\lambda_2)}{[r_1, r_2]\cos\beta_3\sin(\lambda_3-\lambda_2)}, \\[2ex] M''' = \dfrac{[r_2, r_3]}{[r_1, r_2]}\,\dfrac{\sin(\lambda_2-\lambda_1)}{\sin(\lambda_3-\lambda_2)}\,\dfrac{\cos\beta_1}{\cos\beta_3}. \end{cases}$$

207 The Approximation in Olbers' Method The coefficients m', M', m'', M'', m''', M''', except the ratios of the triangles, are given by the observations It is in computing these ratios that the approximations enter in Olbers' method In every practical method which has been devised it has been necessary to make some assumption which is not strictly true In one of the earlier ones it was assumed as a first approximation that in the interval covered by the observations the comet moves in a straight line with uniform speed*

In many cases this is so far from the truth that the method is of little value In Olbers' method it is assumed that the triangular areas bounded by the radii and the chord joining their extremities are proportional to the times between the corresponding observations According to the law of areas the proportion is rigorously true for the sectors, and it is evident from the geometry of the problem that it is very nearly true when the triangles are substituted for the sectors, especially when the intervals between the observations are short The ratio of the triangle $[r_1, r_2]$ to the triangle $[r_2, r_3]$ is more nearly equal to the ratio of the times in which they are described by the radius vector, the more nearly the second observation divides the whole interval into two equal parts The proportion will be more in error when the triangles $[r_1, r_2]$ and $[r_1, r_3]$ are considered The assumption is that

$$\frac{[r_i,\ r_j]}{[r_k,\ r_l]} = \frac{t_j - t_i}{t_l - t_k} \tag{31}$$

In order to get a better idea of the magnitude of the error committed in the assumption, and to have a method of correction when the heliocentric distances have been approximately determined, it will be necessary to develop the analytical expressions for the ratios of the triangles

208 The Ratios of the Triangles The mass of the comet, m, is so small compared to that of the sun as to be absolutely inappreciable, therefore $\sqrt{S+m}=1$, since the mass of the sun is taken as unity

Let

$$\tau = k\,(t - t_0)$$

* Boscovich, in *Pingré Cométographie*, vol II p 308

Then, if the attractions of the planets be neglected, the differential equations which the motion of the comet fulfills are

$$(32)\qquad \begin{cases} \dfrac{d^2x}{d\tau^2} = -\dfrac{x}{r^3}, \\ \dfrac{d^2y}{d\tau^2} = -\dfrac{y}{r^3}, \\ \dfrac{d^2z}{d\tau^2} = -\dfrac{z}{r^3} \end{cases}$$

Suppose the coordinates at the particular time t_0 are x_0, y_0, z_0, $\frac{dx_0}{dt}$, $\frac{dy_0}{dt}$, $\frac{dz_0}{dt}$, at any other time they are functions of these initial coordinates and the interval of time $t - t_0$, or of τ This dependence may be indicated by the equation

$$x = f\left(x_0,\ y_0,\ z_0,\ \frac{dx_0}{d\tau},\ \frac{dy_0}{d\tau},\ \frac{dz_0}{d\tau},\ \tau\right)$$

If the interval of time is sufficiently short this may be expanded by Maclaurin's formula, giving

$$(33)\qquad x = f\left(x_0,\ \ ,\frac{dz_0}{d\tau},\ 0\right) + \frac{\partial f}{\partial\tau}\tau + \frac{\partial^2 f}{\partial\tau^2}\frac{\tau^2}{1\ \ 2} + \frac{\partial^3 f}{\partial\tau^3}\frac{\tau^3}{1\ \ 2\ \ 3} +$$

In the partial derivatives of f, τ is to be put equal to zero after differentiation, therefore

$$(34)\qquad \frac{\partial f}{\partial\tau} = \frac{dx_0}{d\tau}, \qquad \frac{\partial^2 f}{\partial\tau^2} = \frac{d\,x_0}{d\tau^2},$$

From (32) it follows that

$$(35)\qquad \begin{cases} \dfrac{d^2x_0}{d\tau} = -\dfrac{x_0}{r_0^3}, \\ \dfrac{d^3x_0}{d\tau^3} = \dfrac{3x_0}{r_0^4}\dfrac{dr_0}{d\tau} - \dfrac{1}{r_0^3}\dfrac{dx_0}{d\tau}, \\ \dfrac{d^4x_0}{d\tau^4} = x_0\left\{\dfrac{1}{r_0^6} - \dfrac{12}{r_0^5}\left(\dfrac{dr_0}{d\tau}\right) + \dfrac{3}{r_0^4}\left(\dfrac{d\,r_0}{d\tau}\right)\right\} + \dfrac{6}{r_0^4}\dfrac{dr_0}{d\tau}\dfrac{dx_0}{d\tau} \end{cases}$$

Let

$$(36)\qquad \begin{cases} A = 1 - \dfrac{1}{2}\dfrac{\tau^2}{r_0^3} + \dfrac{1}{2}\dfrac{\tau^3}{r_0^4}\left(\dfrac{dr_0}{d\tau}\right) + \dfrac{\tau^4}{24}\left\{\dfrac{1}{r_0^6} - \dfrac{12}{r_0^5}\left(\dfrac{dr_0}{d\tau}\right)^2 + \dfrac{3}{r_0^4}\left(\dfrac{d^2r_0}{d\tau^2}\right)\right\} + \qquad , \\ B = \tau - \dfrac{1}{6}\dfrac{\tau^3}{r_0^3} + \dfrac{1}{4}\dfrac{\tau^4}{r_0^4}\left(\dfrac{dr_0}{d\tau}\right) + \end{cases}$$

As a consequence of (34), (35), and (36), (33) becomes

$$(37)\qquad \begin{cases} x = Ax_0 + B\dfrac{dx_0}{d\tau}, \text{ and similarly} \\[2ex] y = Ay_0 + B\dfrac{dy_0}{d\tau}, \\[2ex] z = Az_0 + B\dfrac{dz_0}{d\tau} \end{cases}$$

The ratios of the triangles $[r_i, r_j]$ are equal to the ratios of their projections upon any plane, therefore

$$(38)\qquad \begin{cases} \dfrac{[r_2, r_3]}{[r_1, r_2]} = \dfrac{x_2y_3 - y_2x_3}{x_1y_2 - y_1x_2}, \\[2ex] \dfrac{[r_1, r_2]}{[r_1, r_3]} = \dfrac{x_1y_2 - y_1x_2}{x_1y_3 - y_1x_3} \end{cases}$$

Let

$$(39)\qquad \tau_3 = k\,(t_2 - t_1), \quad \tau_1 = k\,(t_3 - t_2), \quad \tau_2 = (t_3 - t_1),$$

and suppose that x_2, y_2, z_2, $\dfrac{dx_2}{d\tau}$, $\dfrac{dy_2}{d\tau}$, $\dfrac{dz_2}{d\tau}$, are taken as the zero values for the expansions of the type (37) Since the time from t_2 to t_1 is $-\dfrac{\tau_3}{k}$, it follows that

$$(40)\qquad \begin{cases} x_1 = A_1x_2 + B_1\dfrac{dx_2}{d\tau}, \\[2ex] y_1 = A_1y_2 + B_1\dfrac{dy_2}{d\tau}, \\[2ex] x_3 = A_3x_2 + B_3\dfrac{dx_2}{d\tau}, \\[2ex] y_3 = A_3y_2 + B_3\dfrac{dy_2}{d\tau}, \\[2ex] A_1 = 1 - \dfrac{1}{2}\dfrac{\tau_3^2}{r_2^3} - \dfrac{1}{2}\dfrac{\tau_3^3}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + \dfrac{\tau_3^4}{24}\left\{\dfrac{1}{r_2^6} - \dfrac{12}{r_2^5}\left(\dfrac{dr_2}{d\tau}\right)^2 + \dfrac{3}{r_2^4}\left(\dfrac{d^2r_2}{d\tau^2}\right)\right\} - \quad, \\[2ex] B_1 = -\tau_3 + \dfrac{1}{6}\dfrac{\tau_3^3}{r_2^3} + \dfrac{1}{4}\dfrac{\tau_3^4}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) - \quad, \\[2ex] A_3 = 1 - \dfrac{1}{2}\dfrac{\tau_1^2}{r_2^3} + \dfrac{1}{2}\dfrac{\tau_1^3}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + \dfrac{\tau_1^4}{24}\left\{\dfrac{1}{r_2^6} - \dfrac{12}{r_2^5}\left(\dfrac{dr_2}{d\tau}\right)^2 + \dfrac{3}{r_2^4}\left(\dfrac{d^2r_2}{d\tau^2}\right)\right\} + \quad, \\[2ex] B_3 = \tau_1 - \dfrac{1}{6}\dfrac{\tau_1^3}{r_2^3} + \dfrac{1}{4}\dfrac{\tau_1^4}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + \end{cases}$$

As a consequence of equations (40), equations (38) become

$$\frac{[r_2, r_3]}{[r_1, r_2]} = \frac{B_3\left(x\,\dfrac{dy}{d\tau} - y\,\dfrac{dx_2}{d\tau}\right)}{-B_1\left(x\,\dfrac{dy_2}{d\tau} - y\,\dfrac{dx}{d\tau}\right)} = \frac{\tau_1 - \dfrac{1}{6}\dfrac{\tau_1^3}{r^3} + \dfrac{1}{4}\dfrac{\tau_1^4}{r^4}\left(\dfrac{dr}{d\tau}\right) + }{\tau_3 - \dfrac{1}{6}\dfrac{\tau_3^3}{r_2} - \dfrac{1}{4}\dfrac{\tau_3^4}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + },$$

$$\frac{[r_1, r\ ]}{[r_1, r_3]} = \frac{-B_1\left(x_2\dfrac{dy}{d\tau} - y_2\dfrac{dx}{d\tau}\right)}{(A_1B_3 - B_1A_3)\left(x_2\dfrac{dy}{d\tau} - y\,\dfrac{dx_2}{d\tau}\right)} = \frac{\tau_3 - \dfrac{1}{6}\dfrac{\tau_3^3}{r_2^3} - \dfrac{1}{4}\dfrac{\tau_3^4}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + }{\tau_2 - \dfrac{1}{6}\dfrac{\tau_2^3}{r_2^3} + \dfrac{1}{4}\dfrac{\tau^3(\tau_1 - \tau_3)}{r_2^4}\left(\dfrac{dr_2}{d\tau}\right) + },$$

or, carrying out the indicated divisions,

$$(41)\quad \begin{cases} \dfrac{[r_2, r_3]}{[r_1, r\ ]} = \dfrac{\tau_1}{\tau_3}\left(1 - \dfrac{1}{6}\,\dfrac{\tau_1\ - \tau_3^2}{r_2^3} + \dfrac{1}{4}\,\dfrac{\tau_1^3 + \tau_3}{r^4}\,\dfrac{dr}{d\tau} + \quad\right), \\[2ex] \dfrac{[r_1, r\ ]}{[r_1, r_3]} = \dfrac{\tau_3}{\tau_2}\left(1 + \dfrac{1}{6}\,\dfrac{\tau_2\ - \tau_3^2}{r_2^3} - \dfrac{1}{4}\,\dfrac{\tau_1\,(\tau_1\tau\ - \tau_3^2)}{r_2^4}\,\dfrac{dr}{d\tau} + \quad\right) \end{cases}$$

The higher terms in these series are very small compared to the first when the intervals of time between the observations are short unless the comet is very near the sun, when they may become appreciable If the second observation is exactly midway between the first and third observations, $\tau_1 = \tau_3$ and the second term of the first series vanishes, and the higher terms begin with the third order There is no such reduction in the second series It is likewise clear from geometrical considerations that the error made in retaining only the first terms is less in the first ratio than in the second It will be advisable to choose the observations upon which to base the determination of the elements so that they are as nearly equidistant as possible Instead of neglecting the second term in the second series, and in the first series when the intervals of time are not equal, r_2 may be taken equal to unity and the term included While this will not be the correct value of r_2, it will generally be approximately true, and almost invariably more nearly true than to neglect the second term, which is equivalent to putting r equal to infinity

It will be necessary in an unknown orbit to neglect all of the higher terms, and it is in this and in the possible errors in the second terms that the approximation enters A short digression will now be made to investigate what will be the probable relative magnitude of the third terms Consider the one in the first series Suppose the intervals between the observations are five days Then $\tau_1 = \tau_3 = 5k = \frac{1}{12}$ approximately The comet will usually be bright enough and near

enough to the earth to be seen when $r_2=1$, which may be taken in the absence of any knowledge on the subject Therefore

$$\frac{1}{4}\frac{\tau_1^3+\tau_3^3}{r_2^4}=\frac{1}{3456}$$

The factor $\frac{dr_2}{d\tau}=\frac{1}{k}\frac{dr_2}{dt}$ depends upon the eccentricity and the parameter of the orbit It would be zero in a circular orbit, and greatest if the comet were falling in a straight line toward the sun It was shown in Art 89 that the velocity in a parabolic orbit at a given point is to that in a circular orbit at the same distance from the sun as $\sqrt{2}$ is to 1 The velocity of the earth is about 18 5 miles per second, therefore, neglecting the eccentricity of the earth's orbit,

$$\frac{dr_2}{d\tau}=\frac{1}{k}\frac{dr_2}{dt}=\frac{60\times\sqrt{2}\times 18\ 5\times 60\times 60\times 24}{93{,}000{,}000}=1\ 458,$$

and therefore

$$\frac{1}{4}\frac{\tau_1^3+\tau_3^3}{r^4}\frac{dr_2}{d\tau}=0\ 000422$$

While this number might be considerably increased by r_2 being less than unity, nevertheless it will very seldom be equalled because the comet in general will not be moving directly toward the sun, or even nearly toward it The quantities m', M', m'', M'', m''', M''', may therefore be regarded as being known with sufficient accuracy except when the trigonometrical coefficients become indeterminate

209 Choice of the Linear Equation The three linear equations in ρ_3 and ρ_1, (23), (27), and (29), have been derived from (21) One of them will be used simultaneously with Euler's equation

$$6k(t_3-t_1)=(r_1+r_3+s)^{\frac{3}{2}}\mp(r_1+r_3-s)^{\frac{3}{2}},$$

which can be expressed in terms of ρ_1 and ρ_3, for the determination of these quantities The one of the three is to be chosen in which m and M are the most accurately determined This will depend upon the trigonometrical coefficients

It should be remarked first that m', m'', m''', each contains a factor which is nearly equal to zero It follows from the definition of L_1, and L_3 in (22) that $L_3=-\frac{(t_3-t_2)}{(t_2-t_1)}L_1$ very nearly

Therefore the second factor of m' defined by (26) is nearly

$$\text{(42)}\qquad \frac{[r_2, r_3]}{[r_1, r_2]}L_1+L_3=\left(\frac{t_3-t_2}{t_2-t_1}\right)L_1-\left(\frac{t_3-t_2}{t_2-t_1}\right)L_1=0$$

Since the intervals between the observations are short, $\Lambda_1 - \Lambda$ and $\Lambda_3 - \Lambda_2$ are small angles and the radii of the earth's orbit are nearly constant, therefore m'' defined by (28) becomes approximately

$$(43)\qquad \frac{R_2 \sin\beta \ \cos\beta_3''\{(t_3 - t_2) + (t - t_1) - (t_3 - t_1)\}}{\cos(\lambda - \Lambda)\sin(\beta_3'' - \beta_3)\cos\beta_2 (t - t_1)} = 0$$

The approximation to zero in this case is not, in general, so near as in the former case

The angles $\Lambda_1 - \lambda$, $\Lambda_3 - \lambda$ and $\Lambda - \lambda_2$ which occur in (30) will be approximately equal for short intervals of time, and therefore m''' is a very small quantity The approximation to zero in this case is not, in general, so near as in the former cases Therefore, if no difficulties arise from the other factors equation (23) will be preferable, being the simplest since m' may be entirely neglected

The circumstances under which the trigonometrical coefficients become indeterminate must now be investigated When the comet moves in, or near, a great circle passing through the position of the sun at t_2, then $\beta_3' = \beta_3$ and $\beta_1' = \beta_1$ approximately, and m' and M', given by (26), have the indeterminate forms $m' = \frac{0}{0}$, $M' = \frac{0}{0}$ When the comet is in, or near, opposition to, or conjunction with, the sun at t_2, then I is approximately 90, as can be seen from (25), and m' in (26) has the indeterminate form $m' = \infty \times 0$ When the latitudes β_1, β_2, and β_3 are small, or near 90, m and M' are seen from (24) to have the form $m' = \frac{0}{0}$, $M' = \frac{0}{0}$ In these four cases equation (23) cannot be used

Consider the coefficients of equation (27) The auxiliaries β_1'' and β_3'' depend upon $\cos(\lambda_1 - \Lambda)$ and $\cos(\lambda_3 - \Lambda_2)$ in the same way that β_1' and β_3' depended upon $\sin(\lambda_1 - \Lambda)$ and $\sin(\lambda_3 - \Lambda_2)$ Therefore m'' and M'' are not well determined when the motion of the comet is near a great circle at right angles to the circle passing through the position of the comet and that of the sun at the epoch t_2 Likewise m'' and M'' are not well determined when the comet is near quadrature with respect to the sun at the epoch t_2 When the observed latitudes of the comet are small, or near 90°, the equations for m'' and M'' are again nearly indeterminate In these cases equation (27) cannot be used

Equations (30) are indeterminate when the latitudes are near 90°, or when the motion in longitude is very slow In these two cases equation (29) cannot be used If the three equations (23), (27), and (29) fail it indicates that more observations are needed in order to determine the elements with accuracy

The general rule to follow is this Use (23) if possible because of the smallness of the factor in m' If this equation cannot be used (27)

should next be examined, and (29) should be used only when the others fail It must be borne in mind that an equation should be considered as being inapplicable when it approaches anywhere near indeterminateness

210 Method of Solving the Equations Let $\rho_3 = m + M\rho_1$ represent equation (23), (27), or (29), depending upon the requirement of the problem Then from the work which precedes the following equations are available for the determination of the geocentric or heliocentric distances at the epochs t_1 and t_3

$$
(44)\quad \begin{cases}
\rho_3 = \left(\dfrac{m}{\rho_1} + M\right)\rho_1, \\
6k(t_3 - t_1) = (r_1 + r_3 + s)^{\frac{3}{2}} \mp (r_1 + r_3 - s)^{\frac{3}{2}}, \\
s^2 = (x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2 \\
\quad = \rho_1^2 - 2\rho_1 R_1 \cos\beta_1 \cos(\lambda_1 - \Lambda_1) + R_1^2 \\
\qquad + \rho_3^2 - 2\rho_3 R_3 \cos\beta_3 \cos(\lambda_3 - \Lambda_3) + R_3^2 \\
\qquad - 2\rho_1\rho_3\{\sin\beta_1 \sin\beta_3 + \cos\beta_1 \cos\beta_3 \cos(\lambda_3 - \lambda_1)\} \\
\qquad + 2\rho_1 R_3 \cos\beta_1 \cos(\lambda_1 - \Lambda_3) + 2\rho_3 R_1 \cos\beta_3 \cos(\lambda_3 - \Lambda_1) \\
\qquad - 2R_1 R_3 \cos(\Lambda_3 - \Lambda_1), \\
r_1^2 = \rho_1^2 - 2\rho_1 R_1 \cos\beta_1 \cos(\lambda_1 - \Lambda_1) + R_1^2, \\
r_3^2 = \rho_3^2 - 2\rho_3 R_3 \cos\beta_3 \cos(\lambda_3 - \Lambda_3) + R_3^2
\end{cases}
$$

The first equation was derived from the fact that the comet moves in a plane passing through the center of the sun, and that the law of areas holds It is valid for an orbit with any eccentricity The last three equations are geometrical and hold for all orbits The second equation depends upon the fact that the comet moves in a parabola, at least by assumption The corresponding equation in the case of elliptic and hyperbolic orbits is transcendental, and another method must be used

By means of the first equation ρ_3 can be eliminated from all of the remaining equations It has been shown that m is very small, hence, until an approximate value has been found, it will be sufficient to assume that ρ_1 is equal to unity, and to write $\rho_3 = \left(\dfrac{m}{1} + M\right)\rho_1$ This puts all of the three linear equations (23), (27), and (29) in the same form

The second equation can be rationalized, after which the problem becomes the solution of four simultaneous polynomials, but the degrees of the equations are so high that a direct process is impracticable It

has been found that the most convenient method of obtaining the solution is by successive trials, each one consisting of three steps (*a*) Values of r_1 and r_3 are assumed and s is computed from the second equation (Euler's equation) It is advantageous to start with values as nearly correct as possible When the orbit is entirely unknown it is customary to start with $r_1 = r_3 = 1$, because the chances are that the comet will be somewhere near this distance from the sun before it is discovered If, however, the angular distance of the comet from the sun is much greater than 90°, larger values might be used (*b*) ρ_1 is computed from the third equation from which ρ_3 has been eliminated by the first equation This value of ρ_1 may be used to correct the values used in the first equation if the change is sensible in the quotient $\frac{m}{\rho_1}$ (*c*) With the value of ρ_1 found, r_1 and r_3 are computed from the fourth and fifth equations With the new values of r_1 and r_3, s is again computed from Euler's equation and the whole process is repeated This is continued until the values of r_1 and r_3 started with in Euler's equation are the same as those found by the fourth and fifth equations A very few repetitions are usually sufficient to attain the desired accuracy

211 (*a*) **Solution of Euler's Equation for s** The solution of Euler's equation for s may be reduced to the solution of a cubic equation by the proper transformation For this purpose let

$$\sin\gamma = \frac{s}{r_1 + r_3} \tag{45}$$

The upper sign in Euler's equation is to be taken when the heliocentric motion is less than 180 When the interval of time is short, as it is in practice, the motion is nearly always less than 180 and it is safe to take the upper sign, if this assumption should be wrong, the orbit would not represent the middle observation and the computation would have to be repeated using the other sign

Making the substitution (45), it is found after some reduction that

$$\frac{6k(t_3 - t_1)}{2^{\frac{3}{2}}(r_1 + r_3)^{\frac{3}{2}}} = \frac{3\sin\frac{\gamma}{2}}{\sqrt{2}} - 4\left(\frac{\sin\frac{\gamma}{2}}{\sqrt{2}}\right)^3 \tag{46}$$

This is a cubic equation for the determination of $\sin\frac{\gamma}{2}$, and γ will be taken in the first quadrant To solve the cubic, let

$$\sin\theta = \frac{\sin\frac{\gamma}{2}}{\sqrt{2}}, \tag{47}$$

whence

$$6k(t_3-t_1) \over 2^{\frac{3}{2}}(r_1+r_3)^{\frac{3}{2}} = \sin 3\theta \tag{48}$$

It follows from (47) that $\theta \leqq 30°$ The equations (48), (47), and (45) are to be used in order for the determination of s

Tables have also been constructed which save nearly all the computation Let

$$\left\{\begin{aligned} \eta &= \frac{2k(t_3-t_1)}{(r_1+r_3)^{\frac{3}{2}}}, \\ s &= \frac{2k(t_3-t_1)}{\sqrt{r_1+r_3}}\mu \end{aligned}\right. \tag{49}$$

Table XI in Watson's *Theoretical Astronomy*, and VII in Oppolzer's *Bahnbestimmung*, give the value of $\log \mu$ with the argument η, and it is advantageous to use one of these tables when possible*

212 (*b*) Solution of s^2 for ρ_1 Let

$$\left\{\begin{aligned} &\frac{m}{\rho_1} + M = N, \\ &\rho_3 = N\rho_1 \end{aligned}\right. \tag{50}$$

Substituting this value of ρ_3 in the third equation of (44), it becomes a quadratic in ρ_1 and can be readily solved The transformations introduced by Gauss greatly facilitate the computation The rectangular heliocentric coordinates are expressed in terms of the polar geocentric coordinates by the equations

$$\left\{\begin{aligned} x_3 - x_1 &= \rho_1(N\cos\lambda_3\cos\beta_3 - \cos\lambda_1\cos\beta_1) - R_3\cos\Lambda_3 + R_1\cos\Lambda_1, \\ y_3 - y_1 &= \rho_1(N\sin\lambda_3\cos\beta_3 - \sin\lambda_1\cos\beta_1) - R_3\sin\Lambda_3 + R_1\sin\Lambda_1, \\ z_3 - z_1 &= \rho_1(N\sin\beta_3 - \sin\beta_1) \end{aligned}\right. \tag{51}$$

Then Gauss introduced the auxiliaries g and G by the equations

$$\left\{\begin{aligned} R_3\cos\Lambda_3 - R_1\cos\Lambda_1 &= g\cos G, \\ R_3\sin\Lambda_3 - R_1\sin\Lambda_1 &= g\sin G, \quad g > 0, \end{aligned}\right.$$

or, more conveniently for determining g and G,

$$\left\{\begin{aligned} R_3\cos(\Lambda_3-\Lambda_1) - R_1 &= g\cos(G-\Lambda_1), \\ R_3\sin(\Lambda_3-\Lambda_1) \qquad &= g\sin(G-\Lambda_1) \end{aligned}\right. \tag{52}$$

* These tables are extensions of one computed by Gauss and given in the *Theoria Motus*

Then let

$$\begin{cases} N\cos\lambda_3\cos\beta_3 - \cos\lambda_1\cos\beta_1 = h\cos\zeta\cos H, \\ N\sin\lambda_3\cos\beta_3 - \sin\lambda_1\cos\beta_1 = h\cos\zeta\sin H, \\ N\sin\beta_3 \qquad\quad - \sin\beta_1 \qquad\quad = h\sin\zeta, \end{cases}$$

which determine h, ζ, and H uniquely if it is agreed that

$$h > 0, \quad -90 \; < \zeta < 90$$

The quantity N is computed from (50) in putting ρ_1 equal to unity If the correct value of ρ_1 does not give a sensibly different value for N the computations of h, ζ, and H are made once for all, otherwise they must be repeated with a more nearly correct value of N When m', equation (26), is used this repetition will almost never be necessary For the calculation it is more convenient to rotate the axes through the angle λ_1, and have

$$(53) \quad \begin{cases} h\cos\zeta\cos(H-\lambda_1) = N\cos(\lambda_3-\lambda_1)\cos\beta_3 - \cos\beta_1, \\ h\cos\zeta\sin(H-\lambda_1) = N\sin(\lambda_3-\lambda_1)\cos\beta_3, \\ h\sin\zeta \qquad\qquad\quad = N\sin\beta_3 - \sin\beta_1 \end{cases}$$

Equations (51) become, as a consequence of (52) and (53),

$$(54) \quad \begin{cases} x_3 - x_1 = \rho_1 h\cos\zeta\cos H - g\cos G, \\ y_3 - y_1 = \rho_1 h\cos\zeta\sin H - g\sin G, \\ z_3 - z_1 = \rho_1 h\sin\zeta, \end{cases}$$

and the third equation of (44) becomes

$$(55) \quad s^2 = \rho_1^2 h^2 - 2\rho_1 hg\cos\zeta\cos(G-H) + g$$

Now make the substitution

$$(56) \quad \begin{cases} \cos\phi = \cos\zeta\cos(G-H), \\ \sin\phi\cos Q = \cos\zeta\sin(G-H), \\ \sin\phi\sin Q = \sin\zeta, \qquad 0 < \phi < 180 \;, \end{cases}$$

then (55) becomes

$$(57) \quad \begin{cases} s^2 = (h\rho_1 - g\cos\phi) \; + g \; \sin^2\phi, \text{ whence} \\ \rho_1 = \dfrac{g}{h}\cos\phi \pm \dfrac{1}{h}\sqrt{s^2 - g^2\sin\;\phi} \end{cases}$$

It is necessary to decide whether + or − should be taken before the radical It depends upon whether $h\rho_1 - g\cos\phi$ is greater or less than zero From the first equation of (57) it is easily found that

$$s^2 - g \; = h\rho_1(h\rho_1 - 2g\cos\phi)$$

Therefore, if $s>g$, it follows that $h\rho_1-2g\cos\phi>0$, and therefore $h\rho_1-g\cos\phi>0$ From equation (52) it is found that

$$g^2=R_1^2+R_3^2-2R_1R_3\cos(\Lambda_3-\Lambda_1),$$

that is, g is the chord joining the positions of the sun at the epochs t_3 and t_1 The length of the chord is nearly proportional to the velocity in the orbit for a short time Therefore, neglecting the curvature of the sun's orbit and remembering that its radius is unity, it follows from the formula $V^2=k^2\left(\frac{2}{r}-\frac{1}{a}\right)$ that

$$(58)\qquad \begin{cases} g=k(t_3-t_1), \\ s=k\sqrt{\dfrac{2}{r}}\,(t_3-t_1), \end{cases}$$

where r is the distance of the comet from the sun The limit of the inequality $s>g$ is $s=g$, or from equations (58), $r=2$ Therefore, if the comet is less than twice as far distant from the sun as the earth is at the time of the observation, the + sign must be used in (57) As comets are rarely observed at a greater distance from the sun than this, it will be safe in practice always to use the positive sign If the comet were farther from the sun than this limit it would not be possible to decide directly which sign should be used However, in practice, the expression for ρ_1 is

$$(59)\qquad \rho_1=\frac{g\cos\phi+\sqrt{s^2-g^2\sin^2\phi}}{h}$$

213 (*c*) **Solution for r_1 and r_3** The fourth and fifth equations of (44) give r_1 and r_3 directly, but it will be more convenient in the computation to make a transformation of variables

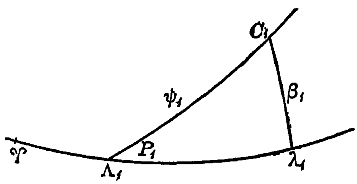

Fig 54

Pass a great circle through the positions of the comet and the sun at the epoch t_2 and denote the various arcs and angles as in the figure

Then it follows from this, and a similar figure for the corresponding positions at t_3, that

$$
(60)\quad \begin{cases} \cos\psi_1 = \cos\beta_1\cos(\lambda_1-\Lambda_1), & \cos\psi_3 = \cos\beta_3\cos(\lambda_3-\Lambda_3), \\ \cos P_1 \sin\psi_1 = \cos\beta_1 \sin(\lambda_1-\Lambda_1), & \cos P_3\sin\psi_3 = \cos\beta_3\sin(\lambda_3-\Lambda_3), \\ \sin P_1\sin\psi_1 = \sin\beta_1, & \sin P_3 \sin\psi_3 = \sin\beta_3 \end{cases}
$$

These equations determine ψ_1 and ψ_3 uniquely, and these auxiliaries are computed once for all The last two equations of (44) become as a consequence of (60)

$$
(61)\quad \begin{cases} r_1 = \sqrt{(\rho_1 - R_1\cos\psi_1)^2 + R_1^2 \sin^2\psi_1}, \\ r_3 = \sqrt{(\rho_3 - R_3\cos\psi_3)^2 + R_3^2\sin^2\psi_3} = \sqrt{(N\rho_1 - R_3\cos\psi_3)^2 + R_3^2\sin^2\psi_3} \end{cases}
$$

Let

$$
(62)\qquad \tan\theta_1 = \frac{\rho_1 - R_1\cos\psi_1}{R_1\sin\psi_1}, \quad \tan\theta_3 = \frac{N\rho_1 - R_3\cos\psi_3}{R_3\sin\psi_3},
$$

then equations (61) become

$$
(63)\qquad \begin{cases} r_1 = R_1\sin\psi_1\sec\theta_1, \\ r_3 = R_3\sin\psi_3\sec\theta_3 \end{cases}
$$

With these values of r_1 and r_3 the steps (a), (b), and (c) are repeated, and the process is continued until the values obtained by (63) are the same as those started with in Euler's equation This will happen after a few trials unless the observations are such that some of the transformations lead to indeterminate equations

214 Differential Corrections The processes above generally may be much shortened by applying differential corrections Let $\sigma = r_1 + r_3$ for abbreviation, and let σ_0 be the original approximate value The values of r_1 and r_3 found by (63) are functions of σ_0 Thus

$$
r_1 + r_3 = f(\sigma_0)
$$

When the correct σ is used on the start the same value will be found at the end, the equation expressing this condition is

$$
(64)\qquad \sigma_0 + \Delta\sigma_0 = f(\sigma_0 + \Delta\sigma_0)
$$

Expanding the right member by Taylor's formula and neglecting terms whose degree is higher than the first, and solving for $\Delta\sigma_0$, it is found that

$$
(65)\qquad \Delta\sigma_0 = \frac{f(\sigma_0) - \sigma_0}{1 - f'(\sigma_0)}
$$

Now

$$
f'(\sigma_0) = \left(\frac{\partial f}{\partial\sigma}\right)_{\sigma=\sigma_0} = \left(\frac{\partial r_1}{\partial\theta_1}\frac{\partial\theta_1}{\partial\rho_1} + \frac{\partial r_3}{\partial\theta_3}\frac{\partial\theta_3}{\partial\rho_1}\right)\frac{\partial\rho_1}{\partial s}\frac{\partial s}{\partial\sigma}
$$

From equations (63), (62), (59), and Euler's equation, it is found that

$$\left\{\begin{aligned} \frac{\partial r_1}{\partial \theta_1} &= R_1 \sin \psi_1 \tan \theta_1 \sec \theta_1, \\ \frac{\partial r_3}{\partial \theta_3} &= R_3 \sin \psi_3 \tan \theta_3 \sec \theta_3, \\ \frac{\partial \theta_1}{\partial \rho_1} &= \frac{\cos^2 \theta_1}{R_1 \sin \psi_1}, \\ \frac{\partial \theta_3}{\partial \rho_1} &= \frac{N \cos^2 \theta_3}{R_3 \sin \psi_3}, \\ \frac{\partial \rho_1}{\partial s} &= \frac{s}{h\sqrt{s^2 - g^2 \sin^2 \phi}}, \\ \frac{\partial s}{\partial \sigma} &= \frac{\sqrt{\sigma_0^2 - s^2} - \sigma_0}{s}, \end{aligned}\right.$$

and therefore

$$f'(\sigma_0) = (\sin \theta_1 + N \sin \theta_3) \frac{(\sqrt{\sigma_0^2 - s^2} - \sigma_0)}{h\sqrt{s^2 - g^2 \sin^2 \phi}} \tag{66}$$

All the quantities entering in this formula have been previously computed Substituting in (65) the correction $\Delta\sigma_0$ is determined since σ_0, the original value, and $f(\sigma_0)$, the computed value, of $r_1 + r_3$ are known If the final value of $r_1 + r_3$ is the same as the original value after the correction defined by (65) has been applied, then this part of the work is finished, and ρ_1, ρ_3, r_1, and r_3 have been found, if it is not, a new correction must be applied and the computation carried through again In computing a second correction it will almost invariably be sufficiently accurate to use the $f'(\sigma_0)$ found in making the first correction

215 Computation of the Heliocentric Coordinates If the polar axes at the two epochs be rotated forward in the plane of the ecliptic through the angles Λ_1 and Λ_3 respectively, the equations relating the heliocentric and geocentric coordinates are

$$\left\{\begin{aligned} r_1 \cos b_1 \cos (l_1 - \Lambda_1) &= \rho_1 \cos \beta_1 \cos (\lambda_1 - \Lambda_1) - R_1, \\ r_1 \cos b_1 \sin (l_1 - \Lambda_1) &= \rho_1 \cos \beta_1 \sin (\lambda_1 - \Lambda_1), \\ r_1 \sin b_1 &= \rho_1 \sin \beta_1, \\ r_3 \cos b_3 \cos (l_3 - \Lambda_3) &= \rho_3 \cos \beta_3 \cos (\lambda_3 - \Lambda_3) - R_3, \\ r_3 \cos b_3 \sin (l_3 - \Lambda_3) &= \rho_3 \cos \beta_3 \sin (\lambda_3 - \Lambda_3), \\ r_3 \sin b_3 &= \rho_3 \sin \beta_3 \end{aligned}\right. \tag{67}$$

The right members of these equations are entirely known, therefore r_1, b_1, l_1, r_3, b_3, and l_3 are uniquely determined But r_1 and r_3 are already known from (61), and their recomputation here will serve as a check upon this part of the work

216 Computation of i and Ω Pass a great circle C_1C_3 through the first and third heliocentric places of the comet The

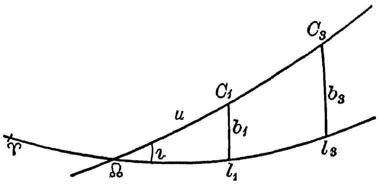

Fig 55

inclination will be less or greater than 90° depending on whether l_3 is greater or less than l_1 In this manner the sign of $\tan i$ is determined Then it follows from the spherical triangles $C_1 \Omega l_1$ and $C_3 \Omega l_3$ that

$$\begin{cases} \tan i \sin(l_1 - \Omega) = \tan b_1, \\ \tan i \sin(l_3 - \Omega) = \tan b_3 \end{cases}$$

But $l_3 - \Omega = (l_3 - l_1) + (l_1 - \Omega)$, therefore these equations become

$$(68) \qquad \begin{cases} \tan i \sin(l_1 - \Omega) = \tan b_1, \\ \tan i \cos(l_1 - \Omega) = \dfrac{\tan b_3 - \tan b_1 \cos(l_3 - l_1)}{\sin(l_3 - l_1)}, \end{cases}$$

which determine i and Ω uniquely, since if $l_3 - l_1 > 0$ the motion is direct and i is less than 90, and conversely

217 Computation of the Argument of the Latitude The argument of the latitude, u, is the longitude from the node From Fig 55 it follows that

$$\begin{cases} \cos(l_1 - \Omega) \cos b_1 = \cos u_1, \\ \sin(l_1 - \Omega) \cos b_1 = \sin u_1 \cos i, \\ \qquad\qquad \sin b_1 = \sin u_1 \sin i, \\ \cos(l_3 - \Omega) \cos b_3 = \cos u_3, \\ \sin(l_3 - \Omega) \cos b_3 = \sin u_3 \cos i, \\ \qquad\qquad \sin b_3 = \sin u_3 \sin i \end{cases}$$

From these equations the following are derived

$$
(69)\quad \begin{cases}
\left.\begin{aligned}
\tan u_1 &= \dfrac{\tan(l_1 - \Omega)}{\cos i}, \\
\tan u_3 &= \dfrac{\tan(l_3 - \Omega)}{\cos i},
\end{aligned}\right\} & \text{(If } i < 45^\circ \text{ or } i > 135^\circ\text{)} \\[2ex]
\left.\begin{aligned}
\tan u_1 &= \dfrac{\tan b_1}{\cos(l_1 - \Omega)\sin i}, \\
\tan u_3 &= \dfrac{\tan b_3}{\cos(l_3 - \Omega)\sin i},
\end{aligned}\right\} & \text{(If } 45^\circ < i < 135^\circ\text{)} \\[2ex]
\left.\begin{aligned}
\sin u_1 &= \dfrac{\sin b_1}{\sin i}, \\
\sin u_3 &= \dfrac{\sin b_3}{\sin i}
\end{aligned}\right\} & \text{(Define the quadrants)}
\end{cases}
$$

Since $u_3 - u_1 = v_3 - v_1$ is the angle between r_1 and r_3 the following equation may be used as a check upon the computation up to this point

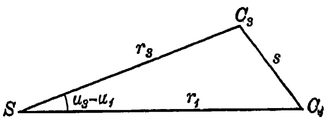

Fig 56

$$
(70)\qquad s^2 = r_1^2 + r_3^2 - 2 r_1 r_3 \cos(u_3 - u_1)
$$

218 Computation of q and π The polar equation of the parabola gives

$$
\begin{cases}
r_1 = \dfrac{p}{1 + \cos v_1} = q \sec^2 \dfrac{v_1}{2}, \\[2ex]
r_3 = \dfrac{p}{1 + \cos v_3} = q \sec^2 \dfrac{v_3}{2},
\end{cases}
$$

whence

$$
\frac{\cos \frac{v_1}{2}}{\sqrt{q}} = \pm \frac{1}{\sqrt{r_1}}, \qquad \frac{\cos \frac{v_3}{2}}{\sqrt{q}} = \pm \frac{1}{\sqrt{r_3}}
$$

If v is counted from -180° to $+180^\circ$ instead of 0° to 360° the ambiguous sign will be avoided Since $v_3 = v_1 + u_3 - u_1$ the last equations give

$$(71)\qquad \begin{cases} \dfrac{\cos\frac{v_1}{2}}{\sqrt{q}} = \dfrac{1}{\sqrt{r_1}}, \\[2ex] \dfrac{\sin\frac{v_1}{2}}{\sqrt{q}} = \dfrac{1}{\sqrt{r_1}} \cot\left(\dfrac{u_3 - u_1}{2}\right) - \dfrac{1}{\sqrt{r_3}} \operatorname{cosec}\left(\dfrac{u_3 - u_1}{2}\right) \end{cases}$$

After q and v_1 have been computed from these equations π is given by

$$(72)\qquad \pi = u_1 - v_1 + \Omega$$

219 Computation of the Time of Perihelion Passage The time of perihelion passage may be computed from the integral of areas,

$$\frac{k(t - T)}{\sqrt{2} q^{\frac{3}{2}}} = \tan\frac{v}{2} + \tfrac{1}{3}\tan^3\frac{v}{2}$$

Let

$$(73)\qquad K = \tan\frac{v}{2} + \tfrac{1}{3}\tan^3\frac{v}{2},$$

then it follows that

$$(74)\qquad T = t_1 - \frac{\sqrt{2}}{k} K_1 q^{\frac{3}{2}} = t_3 - \frac{\sqrt{2}}{k} K_3 q^{\frac{3}{2}}$$

This completes the determination of the elements

220 Computation of an Ephemeris The final test of the elements is to determine whether they are consistent with the observation at t_2, and with others if there are such The theory of computing an ephemeris when the elements are known was given in Chap V It will be sufficient to rewrite the equations here

Define K by the equation

$$(75)\qquad K = \frac{k}{\sqrt{2}} \frac{(t - T)}{q^{\frac{3}{2}}}$$

Then v is given by Table VI * in Watson's *Theoretical Astronomy*, or by Table IV in Oppolzer's *Bahnbestimmung*, or it may be computed by equations (33), Chap V The heliocentric radius vector and argument of latitude are given by the equations

$$(76)\qquad \begin{cases} r = q \sec\dfrac{v}{2}, \\[2ex] u = v + \pi - \Omega \end{cases}$$

* In Watson's *Theoretical Astronomy* K is multiplied by the factor 75

The following equations give the heliocentric ecliptic polar coordinates

$$(77)\qquad \begin{cases} \cos b \cos(l - \Omega) = \cos u, \\ \cos b \sin(l - \Omega) = \sin u \cos i, \\ \sin b \qquad\qquad\quad = \sin u \sin i \end{cases}$$

The geocentric ecliptic coordinates are given by

$$(78)\qquad \begin{cases} \rho \cos\beta \cos(\lambda - \Lambda) = r \cos b \cos(l - \Lambda) + R, \\ \rho \cos\beta \sin(\lambda - \Lambda) = r \cos b \sin(l - \Lambda), \\ \rho \sin\beta \qquad\qquad\quad = r \sin b \end{cases}$$

R and Λ are given in the *Nautical Almanac* for every day of the year

The geocentric equatorial coordinates are found by first computing the auxiliaries A, a, B, b, C, c, from the equations

$$(79)\qquad \begin{cases} \sin a \sin A = \cos \Omega, & \\ \sin a \cos A = -\sin \Omega \cos i, & \sin a > 0, \\ \sin b \sin B = \sin \Omega \cos \epsilon, & \\ \sin b \cos B = \cos \Omega \cos i \cos \epsilon - \sin i \sin \epsilon, & \sin b > 0, \\ \sin c \sin C = \sin \Omega \sin \epsilon, & \\ \sin c \cos C = \cos \Omega \cos i \sin \epsilon + \sin i \cos \epsilon, & \sin c > 0 \end{cases}$$

Then ρ, δ, and α are determined by the equations

$$(80)\qquad \begin{cases} \rho \cos\delta \cos\alpha = r \sin a \sin(A + u) + R \cos L \cos D, \\ \rho \cos\delta \sin\alpha = r \sin b \sin(B + u) + R \sin L \cos D, \\ \rho \sin\delta \qquad\quad = r \sin c \sin(C + u) + R \sin D, \end{cases}$$

where L and D are the right ascension and declination of the sun They are given in the *Nautical Almanac*

Recapitulation of Method and Formulas for the Computation of an Approximate Orbit

221 Preparation of the Observations It will be supposed that the orbit is entirely unknown Three observations are selected so that the interval of time is divided as nearly as possible into equal parts by the second observation The times of the observations are reduced to decimals of a day, Washington or Greenwich mean time,

and the right ascension to degrees, minutes, and seconds The longitudes and latitudes are computed by (Eqs 71, Chap V)

$$\begin{cases} m \sin M^* = \sin\delta, \\ m\cos M = \cos\delta \sin\alpha, \qquad m>0, \\ \cos\beta\cos\lambda = \cos\delta\cos\alpha, \\ \cos\beta\sin\lambda = m\cos(M-\epsilon), \\ \sin\beta \qquad = m\sin(M-\epsilon), \end{cases}$$

where ϵ is the obliquity of the ecliptic, and is to be taken from the *Nautical Almanac* The longitude and the logarithms of the sun's radii vectores at the three epochs are also to be taken from the *Nautical Almanac* Then the following data are available

t_1	λ_1	β_1	Λ_1	$\log R_1$,
t	λ_2	β	Λ,	$\log R_2$,
t_3	λ_3	β_3	Λ_3	$\log R_3$

222 Computation of the Geocentric Distances One relation between the geocentric distances is†

$$(50) \qquad \rho_3 = \left(\frac{m}{\rho_1} + M\right)\rho_1 = N\rho_1$$

Generally m may be put equal to zero The auxiliaries β_1 and β_3' are computed from the equations

$$(25) \qquad \begin{cases} \sin(\lambda - \Lambda) = \tan\beta_2 \cot I, \\ \sin(\lambda_1 - \Lambda) = \tan\beta_1' \cot I, \quad (-90^\circ < \beta_1' < 90), \\ \sin(\lambda_3 - \Lambda) = \tan\beta_3' \cot I, \quad (-90^\circ < \beta_3 < 90^\circ), \end{cases}$$

after which M is given by

$$(26) \qquad M = \frac{t_3 - t_2}{t_2 - t_1} \frac{\sin(\beta_1 - \beta_1')}{\sin(\beta_3' - \beta_3)} \frac{\cos\beta_3'}{\cos\beta_1'}$$

If this expression for M becomes indeterminate the last equation of either (28) or (30) must be used

The auxiliaries g, G, h, H, ζ, $\sin\phi$, $\cos\phi$, $\sin\psi_1$, $\cos\psi_1$, $\sin\psi_3$, $\cos\psi_3$, which are constant throughout the work, are computed from the following equations

$$(52) \qquad \begin{cases} g\cos(G - \Lambda_1) = R_3 \cos(\Lambda_3 - \Lambda_1) - R_1, \\ g\sin(G - \Lambda_1) = R_3\sin(\Lambda_3 - \Lambda_1), \qquad g>0, \end{cases}$$

* The auxiliaries m and M are not to be confused with other quantities having the same designation in other parts of the work, e g as in (50)

† For convenience of reference the formulas are given the same numbers here as in the preceding articles where they are derived

$$(53)\quad \begin{cases} h\cos\zeta\cos(H-\lambda_1)=N\cos(\lambda_3-\lambda_1)\cos\beta_3-\cos\beta_1, \\ h\cos\zeta\sin(H-\lambda_1)=N\sin(\lambda_3-\lambda_1)\cos\beta_3, \\ h\sin\zeta \qquad\qquad\quad =N\sin\beta_3-\sin\beta_1, \qquad h>0,\ -90<\zeta<90, \end{cases}$$

$$(56)\quad \begin{cases} \cos\phi=\cos\zeta\cos(G-H), \\ \cos Q\sin\phi=\cos\zeta\sin(G-H), \\ \sin Q\sin\phi=\sin\zeta, \qquad 0<\phi<180^\circ, \end{cases}$$

$$(60)\quad \begin{cases} \cos\psi_1=\cos\beta_1\cos(\lambda_1-\Lambda_1), & \cos\psi_3=\cos\beta_3\cos(\lambda_3-\Lambda_3), \\ \cos P_1\sin\psi_1=\cos\beta_1\sin(\lambda_1-\Lambda_1), & \cos P_3\sin\psi_3=\cos\beta_3\sin(\lambda_3-\Lambda_3), \\ \sin P_1\sin\psi_1=\sin\beta_1, & \sin P_3\sin\psi_3=\sin\beta_3 \end{cases}$$

Then $\log 2k(t_3-t_1)$ is computed ($\log k = 8\ 235581$)

Equations (44) are solved by successive approximations as follows It is assumed that $\sigma_0 = r_1 + r_3 = 2$ Then η is computed from

$$(49)\qquad \eta=\frac{2k(t_3-t_1)}{(r_1+r_3)^{\frac{3}{2}}}$$

With argument η, $\log\mu$ is taken from Table XI in Watson's *Theoretical Astronomy*, or VII in Oppolzer's *Bahnbestimmung* The following equations are then used in order

$$(49)\qquad s=\frac{2k(t_3-t_1)}{\sqrt{r_1+r_3}}\,\mu,$$

$$(59)\qquad \rho_1=\frac{g\cos\phi+\sqrt{s^2-g^2\sin^2\phi}}{h},$$

$$(62)\qquad \tan\theta_1=\frac{\rho_1-R_1\cos\psi_1}{R_1\sin\psi_1}, \qquad \tan\theta_3=\frac{N\rho_1-R_3\cos\psi_3}{R_3\sin\psi_3},$$

$$(63)\qquad \begin{cases} r_1=R_1\sin\psi_1\sec\theta_1, \\ r_3=R_3\sin\psi_3\sec\theta_3, \end{cases}$$

$$f(\sigma_0)=r_1+r_3=R_1\sin\psi_1\sec\theta_1+R_3\sin\psi_3\sec\theta_3$$

The correction $\Delta\sigma_0$ to be applied to σ_0 is found from

$$(65)\qquad \Delta\sigma_0=\frac{f(\sigma_0)-\sigma_0}{1-f'(\sigma_0)},$$

where

$$(66)\qquad f'(\sigma_0)=(\sin\theta_1+N\sin\theta_3)\,\frac{(\sqrt{\sigma_0^2-s^2}-\sigma_0)}{h\sqrt{s^2-g^2\sin^2\phi}}$$

With the corrected values of σ_0 the computation is repeated, and the process is continued until $\sigma=f(\sigma)$, that is, until the values of r_1 and r_3 are found which satisfy the equation

223 Computation of the Elements The heliocentric coordinates are computed from

$$
(67)\qquad \begin{cases}
r_1 \cos b_1 \cos (l_1 - \Lambda_1) = \rho_1 \cos \beta_1 \cos (\lambda_1 - \Lambda_1) - R_1, \\
r_1 \cos b_1 \sin (l_1 - \Lambda_1) = \rho_1 \cos \beta_1 \sin (\lambda_1 - \Lambda_1), \\
r_1 \sin b_1 \qquad\qquad\qquad = \rho_1 \sin \beta_1, \\
r_3 \cos b_3 \cos (l_3 - \Lambda_3) = \rho_3 \cos \beta_3 \cos (\lambda_3 - \Lambda_3) - R_3, \\
r_3 \cos b_3 \sin (l_3 - \Lambda_3) = \rho_3 \cos \beta_3 \sin (\lambda_3 - \Lambda_3), \\
r_3 \sin b_3 \qquad\qquad\qquad = \rho_3 \sin \beta_3
\end{cases}
$$

The quantities r_1 and r_3 were previously known, and they will serve as a check upon the computation of b_1, b_3, l_1, and l_3 from these equations

The elements Ω and i are given by

$$
(68)\qquad \begin{cases}
\tan i \sin (l_1 - \Omega) = \tan b_1, \\
\tan i \cos (l_1 - \Omega) = \dfrac{\tan b_3 - \tan b_1 \cos (l_3 - l_1)}{\sin (l_3 - l_1)}
\end{cases}
$$

The arguments of the latitude are computed from

$$
(69)\qquad \begin{cases}
\left.\begin{aligned}
\tan u_1 &= \frac{\tan (l_1 - \Omega)}{\cos i}, \\
\tan u_3 &= \frac{\tan (l_3 - \Omega)}{\cos i},
\end{aligned}\right\} & \text{(If } i < 45^\circ \text{ or } i > 135^\circ\text{)} \\
\left.\begin{aligned}
\tan u_1 &= \frac{\tan b_1}{\cos (l_1 - \Omega) \sin i}, \\
\tan u_3 &= \frac{\tan b_3}{\cos (l_3 - \Omega) \sin i},
\end{aligned}\right\} & \text{(If } 45^\circ < i < 135^\circ\text{)} \\
\left.\begin{aligned}
\sin u_1 &= \frac{\sin b_1}{\sin i}, \\
\sin u_3 &= \frac{\sin b_3}{\sin i}
\end{aligned}\right\} & \text{(Define the quadrants)}
\end{cases}
$$

The control formula

$$
(70)\qquad s^2 = r_1 \;\; + r_3^2 - 2 r_1 r_3 \cos (u_3 - u_1)
$$

may be applied

The equations

$$
(71)\qquad \begin{cases}
\dfrac{\sin \frac{v_1}{2}}{\sqrt{q}} = \dfrac{1}{\sqrt{r_1}} \cot \left(\dfrac{u_3 - u_1}{2}\right) - \dfrac{1}{\sqrt{r_3}} \operatorname{cosec} \left(\dfrac{u_3 - u_1}{2}\right), \\
\dfrac{\cos \frac{v_1}{2}}{\sqrt{q}} = \dfrac{1}{\sqrt{r_1}},
\end{cases}
$$

give v_1 and q. Then

$$(72) \qquad \begin{cases} \omega = u_1 - v_1, \\ \pi = \omega + \Omega. \end{cases}$$

The time of perihelion passage is computed from the equation

$$\frac{k(t-T)}{\sqrt{2}q^{\frac{3}{2}}} = \tan\frac{v}{2} + \tfrac{1}{3}\tan^3\frac{v}{2}.$$

With the argument v Barker's table (VI, in Watson; IV, in Oppolzer) gives the value of $K = \tan\frac{v}{2} + \frac{1}{3}\tan^3\frac{v}{2}$.* Then

$$(74) \qquad T = t_1 - \frac{\sqrt{2}q^{\frac{3}{2}}}{k}K_1 = t_3 - \frac{\sqrt{2}q^{\frac{3}{2}}}{k}K_3.$$

224. Comparison with Other Observations. The theoretical position at the time of the second of the three observations used in finding the elements is found by applying equations (75)—(78) inclusive. If other observations have been made the theoretical positions may be compared with them, and the greater the interval of time which they cover, the better is the test of the accuracy of the elements. The observations are usually given in right ascension and declination so that it is necessary to apply equations (79) and (80) before the direct comparisons can be made.

225. Corrections of the Elements. In the computation of the first orbit several approximations are introduced which render the elements obtained more or less incorrect. After the first orbit has been computed the approximations may be so much improved that the errors still committed are entirely insensible.

Suppose the approximate orbit has been computed. The geocentric distances are now known with considerable accuracy, and the corrections for parallax may be employed (see Art. 194). Likewise the reduction of the time may be made (see Art. 196). The correction for aberration, and the reduction to the mean equinox should now be made, if they were omitted in the first computation (see Arts. 197, 198). The computation proceeds precisely as in the first case, except that the latitude of the sun should not be neglected, up to the expressions for the ratios of the triangles, equations (41), when the terms of higher order should be included if they are appreciable. These ratios are now subject to no sensible errors. Therefore N, equation (50), is known within the limits of observation and the tables used. The remaining work is precisely as in the first computation, and the elements obtained

* In Watson's *Theoretical Astronomy* K is multiplied by the factor 75.

cannot be improved by this method However, the first and third observations have been given especial prominence in the work, while others than the three have not been used at all If the middle observation is to have more influence on the result, or if all of the observations are to impose conditions upon the elements, different methods must be employed Since the elements are known with a considerable degree of approximation, the further improvements may be made by differential corrections The condition will first be imposed that the middle observation shall be as nearly satisfied as possible, and then that any number of observations shall be as nearly satisfied as possible

226 Variation of One Geocentric Distance From the manner in which the elements are determined the computed first and third positions will exactly agree with the observed positions, but there will generally be small disagreements between theory and observation in the case of the middle position These discordancies arise from the fact that the orbit may not be strictly a parabola, from the errors in the observations, and from the accumulated errors in the use of logarithmic tables The problem now is to vary the elements, keeping the hypothesis that the orbit is a parabola, so that all three observations shall be exactly satisfied if possible But the elements were derived from the two positions of the comet at t_1 and t_3, that is, from λ_1, β_1, ρ_1, λ_3, β_3, and ρ_3 (Arts 215—219)

Instead of varying the elements these quantities upon which the elements depend may be varied But if the first and third observations are to be fulfilled the only quantities which can be changed are ρ_1 and ρ_3 Since there are but five elements in a parabolic orbit, there must be one relation among the six quantities λ_1, β_1, ρ_1, λ_3, β_3, ρ_3, therefore ρ_1 and ρ_3 cannot be varied independently if the orbit is to remain a parabola

Let $\lambda_2^{(0)}$ and $\beta_2^{(0)}$ represent the computed coordinates at the time t They are functions of the elements, or of the coordinates at the epochs t_1 and t_3 Since only one of the six coordinates, as ρ_1, is to be regarded as variable, $\lambda^{(0)}$ and $\beta_2^{(0)}$ may be written as functions of this quantity, as

$$(81) \qquad \begin{cases} \lambda_2^{(0)} = \phi(\rho_1), \\ \beta_2^{(0)} = \psi(\rho_1) \end{cases}$$

Let $\Delta\rho_1$ be the small correction to be applied to ρ_1 It must be such that it fulfills as nearly as possible the relations

$$\lambda = \phi(\rho_1 + \Delta\rho_1),$$
$$\beta_2 = \psi(\rho_1 + \Delta\rho_1)$$

Developing the right members of these equations by Taylor's formula, and neglecting terms of the second and higher powers in $\Delta\rho_1$, they give

$$(82)\qquad \begin{cases} \Delta\rho_1 = \dfrac{\lambda_2 - \lambda_2^{(0)}}{\phi'(\rho_1)}, \\[2ex] \Delta\rho_1 = \dfrac{\beta_2 - \beta_2^{(0)}}{\psi'(\rho_1)} \end{cases}$$

The arithmetical mean of these two values of $\Delta\rho_1$ may be taken as the best correction to be applied to ρ_1, or the two equations may be given unequal weights if there is any reason for doing so The right members of these equations are all known except $\phi'(\rho_1)$ and $\psi'(\rho_1)$ These functions might be computed by direct processes but it is simpler to obtain them indirectly Take an arbitrary small variation $\Delta'\rho_1$ and compute

$$(83)\qquad \begin{cases} \lambda_2^{(1)} = \phi(\rho_1 + \Delta'\rho_1), \\ \beta_2^{(1)} = \psi(\rho_1 + \Delta'\rho_1) \end{cases}$$

The computation of $\lambda_2^{(1)}$ and $\beta_2^{(1)}$ is similar to that of $\lambda_2^{(0)}$ and $\beta_2^{(0)}$, and no new formulas have to be used Expanding equations (83) and solving, it is found that

$$(84)\qquad \begin{cases} \phi'(\rho_1) = \dfrac{\lambda_2^{(1)} - \lambda_2^{(0)}}{\Delta'\rho_1}, \\[2ex] \psi'(\rho_1) = \dfrac{\beta_2^{(1)} - \beta_2^{(0)}}{\Delta'\rho_1}, \end{cases}$$

the right members of which are entirely known

In choosing the arbitrary variation $\Delta'\rho_1$ care should be taken that it is large enough so that equations (84) do not become numerically indeterminate, and small enough so that the higher terms in the expansions, which are neglected, do not sensibly affect the results

An observation which is not near those used in the computation is a better test of the accuracy of the elements If the difference in the computed and observed position is much, the elements may frequently be advantageously corrected by varying one geocentric distance in the manner just explained

227 Variation of the Elements When many observations of the comet have been made, in general no set of elements can be found exactly satisfying all of them The methods which have been developed derive such elements that two observations always must be exactly satisfied and if one of the geocentric distances is varied this still remains true There is no *à priori* reason why a particular two

should be satisfied rather than any others, and accordingly a method will be developed of satisfying all of them as nearly as possible

Suppose Ω_0, i_0, π_0, q_0 and T_0 are the approximate values of the elements which are supposed to be so nearly correct that the residuals are very small Suppose there are n complete observations made at the epochs t_1, , t_n Let λ_1, , λ_n, β_1, , β_n be the observed coordinates, and $\lambda_1^{(0)}$, , $\lambda_n^{(0)}$, $\beta_1^{(0)}$, , $\beta_n^{(0)}$ those which are computed from the approximate elements Then

$$(85)\qquad \begin{cases} \lambda_j^{(0)} = f(\Omega_0, i_0, \pi_0, q_0, T_0, t_j), \\ \beta_j^{(0)} = g(\Omega_0, i_0, \pi_0, q_0, T_0, t_j), \qquad (j=1, \quad, n) \end{cases}$$

Suppose the values of the elements which most nearly satisfy all of the observations are $\Omega_0 + \Delta\Omega_0$, $i_0 + \Delta i_0$, $\pi_0 + \Delta\pi_0$, $q_0 + \Delta q_0$, $T_0 + \Delta T_0$ They must fulfill as nearly as possible the equations

$$(86)\qquad \begin{cases} \lambda_j = f(\Omega_0 + \Delta\Omega_0, i_0 + \Delta i_0, \quad, T_0 + \Delta T_0, t_j), \\ \beta_j = g(\Omega_0 + \Delta\Omega_0, i_0 + \Delta i_0, \quad, T_0 + \Delta T_0, t_j), \quad (j=1, \quad, n) \end{cases}$$

In order to determine the corrections expand the right members of these equations by Taylor's formula and neglect terms of the second and higher degrees Then the equations may be written

$$(87)\qquad \begin{cases} \lambda_j - \lambda_j^{(0)} = \dfrac{\partial f}{\partial \Omega}\Delta\Omega_0 + \dfrac{\partial f}{\partial i}\Delta i_0 + \dfrac{\partial f}{\partial \pi}\Delta\pi_0 + \dfrac{\partial f}{\partial q}\Delta q_0 + \dfrac{\partial f}{\partial T}\Delta T_0, \\ \beta_j - \beta_j^{(0)} = \dfrac{\partial g}{\partial \Omega}\Delta\Omega_0 + \dfrac{\partial g}{\partial i}\Delta i_0 + \dfrac{\partial g}{\partial \pi}\Delta\pi_0 + \dfrac{\partial g}{\partial q}\Delta q_0 + \dfrac{\partial g}{\partial T}\Delta T_0 \end{cases}$$

The partial derivatives of f and g with respect to the elements can be computed by direct processes, but, as in the case of the variation of the geocentric distance, it is simpler to obtain them indirectly Take an arbitrary small $\Delta'\Omega$ and $\Delta' i = \Delta'\pi = \Delta' q = \Delta' T = 0$ and compute $\lambda_j^{(1)}$ and $\beta_j^{(1)}$ Then the partial derivatives with respect to Ω are given by

$$(88)\qquad \begin{cases} \dfrac{\partial f}{\partial \Omega} = \dfrac{\lambda_j^{(1)} - \lambda_j^{(0)}}{\Delta\Omega}, \\ \dfrac{\partial g}{\partial \Omega} = \dfrac{\beta_j^{(1)} - \beta_j^{(0)}}{\Delta'\Omega} \end{cases}$$

The partial derivatives with respect to the other elements are to be found in a similar manner by varying the respective elements separately

Then equations (87) form a linear system of $2n$ equations with known coefficients for the determination of the five unknowns $\Delta\Omega$, Δi, $\Delta\pi$, Δq, and ΔT If $2n$ is greater than five there is not in general a

unique solution, but the most probable values may be found by the Method of Least Squares In accordance with this method each term of each equation in the whole set is multiplied by the coefficient of $\Delta \Omega$ which occurs in it, and the results are added This gives one linear equation The process is repeated with each unknown, giving in this manner five linear equations with five unknowns, which may be solved by the ordinary processes If the corrections found are large the Least Square adjustment should be repeated, but this does not frequently happen in practice

If the disagreements remaining after these corrections are larger than the errors of observation it means that the orbit is not a parabola or that the perturbations are sensible The change to an ellipse or hyperbola with eccentricity near unity may be made by including in the equations above a term depending upon the eccentricity with $e_0 = 1$ The chances are infinitely small that any orbit should be exactly a parabola, but experience shows that most of the comets' orbits are sensibly parabolic

If a comet should pass near enough to a planet so that its elements were sensibly perturbed they should be computed when it is far from the planet, and the perturbations of them while it is passing near the planet calculated by mechanical quadratures If this brings theory and observation into accord, the elements are to be regarded as satisfactory, if not, differential corrections are to be applied, and their effects included in the calculation of the perturbations

XXVII PROBLEMS

1 Suppose two observations of a comet are given very near together, and another at some distance The first two can be regarded as determining a_1, δ_1, $\frac{da_1}{dt}$, $\frac{d\delta_1}{dt}$ and the third gives a_3, δ_3 Show that in this case a general algebraic solution, similar to that of Art 201, is possible

2 In Olbers' Method the longitude and latitude of the comet are used, show how the equations would be less convenient if the right ascension and declination were used Develop the method in full Would m in general be so small that it could be neglected?

3 The observations give six coordinates a_1, δ_1, a_2, δ_2, a_3 and δ_3, which impose six conditions on the elements Only five conditions need to be used to determine the five elements, where in the work has one condition been omitted?

4 Suppose only the longitude of the comet were known at the time of the middle observation, how would equations (23), (27), and (29) be changed?

5 Suppose two complete observations of the apparent position of a comet, and the velocity in the line of sight at one of them are given, show how the elements of the parabolic orbit can be found

6 Suppose the comet moves in the plane of the ecliptic, how many elements are there to be determined and how many observations of longitude must be given?

7 Show that

$$\frac{[r_2, r_3]}{[r_1, r_3]} = \frac{\tau_1}{\tau_2}\left\{1 + \frac{1}{6}\frac{(\tau_2^2 - \tau_1^2)}{r_2^3} + \frac{1}{4}\frac{\tau_3(\tau_2\tau_3 - \tau_1^2)}{r_2^4}\frac{dr_2}{d\tau} + \quad\right\}$$

8 Prove that when a comet moves in a parabola the component of its velocity along the radius is a maximum when it is 90° from the perihelion point, and that when the perihelion distance is unity, and the distance between the observations is five days, the maximum value of $\frac{1}{4}\frac{\tau_1^3 + \tau_3^3}{r_2^4}\frac{dr_2}{d\tau}$ is 0 0002107

9 Find the geometrical meaning of g, G, h, ζ, H, ϕ, and Q introduced in (52), (53), and (56)

10 Take observations of some new comet from the Journals and make the complete computation of the orbit

HISTORICAL SKETCH AND BIBLIOGRAPHY

The first method of finding the orbit of a comet from three observations was devised by Newton, and is given in the *Principia*, Book III, Prop XLI The solution depends upon a graphical construction, which, by successive approximations, leads to the elements Newton seems to have had trouble with the problem, for he says, "This being a problem of very great difficulty, I tried many methods of resolving it"

Little was done on the theory of orbits after Newton until the time of Euler, since the time of Euler nearly every astronomer of note has contributed more or less to the subject Only the more important contributions can be mentioned here

The earlier methods were, for the most part, based upon one or the other of two assumptions, which are only approximately true, viz that in the interval $t_3 - t_1$ the comet describes a straight line with uniform velocity, or

that the radius vector at the time of the middle observation divides the chord joining the end positions into segments which are proportional to the intervals between the observations

Euler devised a very impractical method in 1740, making use of the second assumption Lagrange published two methods in 1778 (*Mem Berlin Acad Sci*) and one in 1783 (*Berliner Ast Jahrbuch*, or *Coll Works*, VII.) His solutions depended upon an equation of the sixth, seventh, or eighth degree Lambert published a method in 1779 (*Berliner Ast Jahr*) involving the solution of an equation of the sixth degree Laplace published an important method in 1780 (*Mém de l'Acad Roy des Sciences de Paris*, Book II, No 32, of *Méc Cél*) This method is perhaps the most important of the earlier ones Its essential features and points of contact with the method now in use are clearly indicated in Poincaré's preface to Tisserand's *Leçons* Dr Olbers devised his method of computing parabolic orbits in 1797, which has been unsurpassed up to the present time (*Abh Weimar* and *Berliner Ast Jahr* for 1809 and 1833) Gauss developed his method for elliptic orbits in the latter part of 1801, and it was published in 1802 (also in *Theoria Motus*, or *Coll Works*, vol VI) The work of Gauss, which has become classical, was stimulated by the discovery of Ceres Jan 1, 1801 If the orbits of such small bodies as the asteroids could not be computed from a few observations it would be difficult to recover them after having been lost for a time while they were in conjunction with the sun

Among writers on the subject of orbits in some of its various phases may be mentioned Bessel, 1815 (*Abh* III), Lubbock, 1830 (*Monthly Notices*), Cauchy, 1846, '7, '8 (*Comptes Rendus*, XXIII, XXV, XXVI), Encke, 1849 (*Abh.*), Villarceau, 1857 (*Annales de l'Obs de Paris*, vol III), Oppolzer, 1878 (*Astronomische Nachrichten*, XCII), Leuschner, 1897 (Dissertation, by Meyer and Muller, Berlin), Moulton, 1899 (*Astrophysical Journal*, June)

The books to be consulted are Olbers' *Abhandlung uber die leichteste und bequemste Methode die Bahn eines Cometen zu berechnen*, edited by Dr J G Galle, published by Voigt and Gunther, Leipzig, 1864, Gauss' *Theoria Motus*, Watson's *Theoretical Astronomy*, Oppolzer's *Bahnbestimmung*, Klinkerfues' *Theoretische Astronomie*, new edition, and Tisserand's *Leçons sur la Détermination des Orbites*

CHAPTER XI

THEORY OF THE DETERMINATION OF THE ELEMENTS OF ELLIPTIC ORBITS

228 Fundamental Equations in the Problem The elements of an elliptic orbit can be determined, as will be shown later in this chapter*, when two positions of the moving body are known, together with the time it has taken it to pass from one to the other, or when three positions of the body are known, with the time of one observation The preliminary problem is, therefore, to determine the geocentric distances at the epochs of the three observations

Equations (21) of Chap X were derived without making any assumption regarding the species of conic in which the body moves, and are valid for elliptic as well as parabolic orbits These equations are

$$
(1)\quad \begin{cases}
\dfrac{[r_2, r_3]}{[r_1, r_3]}(\rho_1 \cos\beta_1 \cos\lambda_1 - R_1 \cos\Lambda_1) \\
\qquad + \dfrac{[r_1, r_2]}{[r_1, r_3]}(\rho_3 \cos\beta_3 \cos\lambda_3 - R_3 \cos\Lambda_3) = \rho_2 \cos\beta_2 \cos\lambda_2 - R_2 \cos\Lambda_2, \\
\dfrac{[r_2, r_3]}{[r_1, r_3]}(\rho_1 \cos\beta_1 \sin\lambda_1 - R_1 \sin\Lambda_1) \\
\qquad + \dfrac{[r_1, r_2]}{[r_1, r_3]}(\rho_3 \cos\beta_3 \sin\lambda_3 - R_3 \sin\Lambda_3) = \rho_2 \cos\beta_2 \sin\lambda_2 - R_2 \sin\Lambda_2, \\
\dfrac{[r_2, r_3]}{[r_1, r_3]}\rho_1 \sin\beta_1 + \dfrac{[r_1, r_2]}{[r_1, r_3]}\rho_3 \sin\beta_3 \quad = \rho_2 \sin\beta_2
\end{cases}
$$

* Arts 230—235, and 236

These equations involve the unknowns ρ_1, ρ_2, and ρ_3 linearly, and consequently they would define these quantities if the ratios of the triangles were exactly known Suppose for the moment that the ratios of the triangles are known, then it is found from equations (1) that

$$(2) \qquad \Delta\rho_2 = \frac{[r_2, r_3]}{[r_1, r_3]} A + \frac{[r_1, r_2]}{[r_1, r_3]} B + C,$$

where

$$(3) \quad \begin{cases} \Delta = -\sin\beta_1 \cos\beta_2 \cos\beta_3 \sin(\lambda_3 - \lambda_2) \\ \quad + \sin\beta_2 \cos\beta_1 \cos\beta_3 \sin(\lambda_3 - \lambda_1) - \sin\beta_3 \cos\beta_1 \cos\beta_2 \sin(\lambda_2 - \lambda_1), \\ A = \quad R_1 \{\sin\beta_1 \cos\beta_3 \sin(\lambda_3 - \Lambda_1) - \cos\beta_1 \sin\beta_3 \sin(\lambda_1 - \Lambda_1)\}, \\ B = \quad R_3 \{\sin\beta_1 \cos\beta_3 \sin(\lambda_3 - \Lambda_3) - \cos\beta_1 \sin\beta_3 \sin(\lambda_1 - \Lambda_3)\}, \\ C = -R_2 \{\sin\beta_1 \cos\beta_3 \sin(\lambda_3 - \Lambda_2) - \cos\beta_1 \sin\beta_3 \sin(\lambda_1 - \Lambda_2)\} \end{cases}$$

Equations (1) may be solved in the three different ways in which they may be taken in pairs, giving ρ_1 and ρ_3 in terms of ρ_2 and known quantities and the ratios of the triangles The expressions to be used in any particular case are those in which there is the least approach to indeterminateness in the various terms The three sets of equations are respectively

$$(4) \quad \begin{cases} \rho_1 \dfrac{[r_2, r_3]}{[r_1, r_3]} \cos\beta_1 \sin(\lambda_3 - \lambda_1) = \rho_2 \cos\beta_2 \sin(\lambda_3 - \lambda_2) \\ \quad + \dfrac{[r_2, r_3]}{[r_1, r_3]} R_1 \sin(\lambda_3 - \Lambda_1) + \dfrac{[r_1, r_2]}{[r_1, r_3]} R_3 \sin(\lambda_3 - \Lambda_3) - R_2 \sin(\lambda_3 - \Lambda_2), \\ \rho_3 \dfrac{[r_1, r_2]}{[r_1, r_3]} \cos\beta_3 \sin(\lambda_1 - \lambda_3) = \rho_2 \cos\beta_2 \sin(\lambda_1 - \lambda_2) \\ \quad + \dfrac{[r_2, r_3]}{[r_1, r_3]} R_1 \sin(\lambda_1 - \Lambda_1) + \dfrac{[r_1, r_2]}{[r_1, r_3]} R_3 \sin(\lambda_1 - \Lambda_3) - R_2 \sin(\lambda_1 - \Lambda_2), \end{cases}$$

$$(5) \quad \begin{cases} \rho_1 \dfrac{[r_2, r_3]}{[r_1, r_3]} (\cos\beta_1 \sin\beta_3 \cos\lambda_1 - \sin\beta_1 \cos\beta_3 \cos\lambda_3) \\ = \rho_2 (\cos\beta_2 \sin\beta_3 \cos\lambda_2 - \sin\beta_2 \cos\beta_3 \cos\lambda_3) + \dfrac{[r_2, r_3]}{[r_1, r_3]} R_1 \cos\Lambda_1 \sin\beta_3 \\ \quad + \dfrac{[r_1, r_2]}{[r_1, r_3]} R_3 \cos\Lambda_3 \sin\beta_3 - R_2 \cos\Lambda_2 \sin\beta_3, \\ \rho_3 \dfrac{[r_1, r_2]}{[r_1, r_3]} (\cos\beta_3 \sin\beta_1 \cos\lambda_3 - \sin\beta_3 \cos\beta_1 \cos\lambda_1) \\ = \rho_2 (\cos\beta_2 \sin\beta_1 \cos\lambda_2 - \sin\beta_2 \cos\beta_1 \cos\lambda_1) + \dfrac{[r_2, r_3]}{[r_1, r_3]} R_1 \cos\Lambda_1 \sin\beta_1 \\ \quad + \dfrac{[r_1, r_2]}{[r_1, r_3]} R_3 \cos\Lambda_3 \sin\beta_1 - R_2 \cos\Lambda_2 \sin\beta_1, \end{cases}$$

$$
(6)\quad \left\{
\begin{aligned}
&\rho_1 \frac{[r_2, r_3]}{[r_1, r_3]}(\cos\beta_1 \sin\beta_3 \sin\lambda_1 - \sin\beta_1 \cos\beta_3 \sin\lambda_3)\\
&= \rho_2(\cos\beta_2 \sin\beta_3 \sin\lambda_2 - \sin\beta_2 \cos\beta_3 \sin\lambda_3) + \frac{[r_2, r_3]}{[r_1, r_3]} R_1 \sin\Lambda_1 \sin\beta_3\\
&+ \frac{[r_1, r_2]}{[r_1, r_3]} R_3 \sin\Lambda_3 \sin\beta_3 - R_2 \sin\Lambda_2 \sin\beta_3\\
&\rho_3 \frac{[r_1, r_2]}{[r_1, r_3]}(\cos\beta_3 \sin\beta_1 \sin\lambda_3 - \sin\beta_3 \cos\beta_1 \sin\lambda_1)\\
&= \rho_2(\cos\beta_2 \sin\beta_1 \sin\lambda_2 - \sin\beta_2 \cos\beta_1 \sin\lambda_1) + \frac{[r_2, r_3]}{[r_1, r_3]} R_1 \sin\Lambda_1 \sin\beta_1\\
&+ \frac{[r_1, r_2]}{[r_1, r_3]} R_3 \sin\Lambda_3 \sin\beta_1 - R_2 \sin\Lambda_2 \sin\beta_1
\end{aligned}
\right.
$$

Any of the last three pairs of equations defines the geocentric distances, ρ_1 and ρ_3, when ρ_2 is known. The first pair gives practically indeterminate values when the apparent motion in longitude is very small, or when the body is near the pole of the ecliptic. The second pair gives practically indeterminate values when the latitudes are very small, or very nearly 90°. The third pair fails for the same conditions.

Equation (2) apparently furnishes ρ_2 but it must be remembered that the ratios of the triangles are known only approximately, and if these approximate expressions are used, the value of ρ_2 defined in this way may be, as will be shown, not even approximately correct. The reason is the following: Δ is very small, and since ρ_2 is finite the right member must be small also. Now it happens that the known parts of the ratios of the triangles enter in such a way that they mutually cancel until their remainder is as small as, or even smaller than, the first terms of the unknown parts; hence the right member of (2) must be regarded as being essentially unknown.

To exhibit the orders of the members equation (2) must be developed explicitly. As in equations (41), Chapter X, it is found that

$$
(7)\quad \left\{
\begin{aligned}
\frac{[r_2, r_3]}{[r_1, r_3]} &= \frac{\tau_1}{\tau_2}\left\{1 + \frac{1}{6}\frac{\tau_2^2 - \tau_1^2}{r_2^3} + \frac{1}{4}\frac{\tau_3(\tau_2\tau_3 - \tau_1^2)}{r_2^4}\frac{dr_2}{d\tau} + \cdots\right\},\\
\frac{[r_1, r_2]}{[r_1, r_3]} &= \frac{\tau_3}{\tau_2}\left\{1 + \frac{1}{6}\frac{\tau_2^2 - \tau_3^2}{r_2^3} - \frac{1}{4}\frac{\tau_1(\tau_2\tau_1 - \tau_3^2)}{r_2^4}\frac{dr_2}{d\tau} + \cdots\right\}
\end{aligned}
\right.
$$

As a consequence of these expressions equation (2) may be written

$$
(8)\qquad \Delta\rho_2 = P + \frac{Q}{r_2^3},
$$

where

$$(9)\quad \begin{cases} P = A\dfrac{\tau_1}{\tau_2} + B\dfrac{\tau_3}{\tau_2} + C + \dfrac{\tau_1\tau_3}{4\tau_2}\left\{A\dfrac{(\tau_2\tau_3 - \tau_1^2)}{r_2^4}\dfrac{dr_2}{d\tau}\right. \\ \qquad\qquad\qquad\qquad \left. - \dfrac{B(\tau_2\tau_1 - \tau_3^2)}{r_2^4}\dfrac{dr_2}{d\tau}\right\} + \text{higher terms}, \\ Q = \dfrac{A}{6}\dfrac{\tau_1}{\tau_2}(\tau_2^2 - \tau_1^2) + \dfrac{B}{6}\dfrac{\tau_3}{\tau_2}(\tau_2^2 - \tau_3^2) \end{cases}$$

It will now be shown that P is of the same order of magnitude as Q, which is the coefficient of the unknown $\dfrac{1}{r_2^3}$ Let τ_1, τ_2, and τ_3 be taken as small quantities of the first order The coefficients Δ, A, B, and C may be expanded in terms of these quantities For this purpose write

$$\begin{cases} \lambda_1 = \phi(t_1) = \phi[t_2 - (t_2 - t_1)], \\ \lambda_3 = \phi(t_3) = \phi[t_2 + (t_3 - t_2)], \end{cases}$$

which give, upon expansion,

$$\begin{cases} \lambda_1 = \phi(t_2) - \phi'(t_2)(t_2 - t_1) + \quad = \lambda_2 - \tau_3\dfrac{d\lambda_2}{d\tau} + \dfrac{\tau_3^2}{2}\dfrac{d^2\lambda_2}{d\tau^2} - \quad, \\ \lambda_3 = \phi(t_2) + \phi'(t_2)(t_3 - t_2) + \quad = \lambda_2 + \tau_1\dfrac{d\lambda_2}{d\tau} + \dfrac{\tau_1^2}{2}\dfrac{d^2\lambda_2}{d\tau^2} + \quad, \end{cases}$$

and similar expressions for β_1, β_3, R_1, and R_3 The derivatives $\dfrac{d\lambda_2}{d\tau}$ and $\dfrac{d^2\lambda_2}{d\tau^2}$ will be at the most of order zero, and the second and third terms are, therefore, of the first and second order respectively Consequently

$$\begin{aligned} \sin\lambda_1 &= \sin\lambda_2\cos\left(-\tau_3\frac{d\lambda_2}{d\tau} + \frac{\tau_3^2}{2}\frac{d^2\lambda_2}{d\tau^2} - \quad\right) \\ &\quad + \cos\lambda_2\sin\left(-\tau_3\frac{d\lambda_2}{d\tau} + \frac{\tau_3^2}{2}\frac{d^2\lambda_2}{d\tau^2} - \quad\right) \\ &= \sin\lambda_2 - \tau_3\cos\lambda_2\frac{d\lambda_2}{d\tau} + \tfrac{1}{2}\tau_3^2\left[\cos\lambda_2\frac{d^2\lambda_2}{d\tau^2} - \sin\lambda_2\left(\frac{d\lambda_2}{d\tau}\right)^2\right] + \quad, \\ \sin\lambda_3 &= \sin\lambda_2 + \tau_1\cos\lambda_2\frac{d\lambda_2}{d\tau} + \tfrac{1}{2}\tau_1^2\left[\cos\lambda_2\frac{d^2\lambda_2}{d\tau^2} - \sin\lambda_2\left(\frac{d\lambda_2}{d\tau}\right)\right] + \quad, \\ \cos\lambda_1 &= \cos\lambda_2 + \tau_3\sin\lambda_2\frac{d\lambda_2}{d\tau} - \tfrac{1}{2}\tau_3^2\left[\sin\lambda_2\frac{d^2\lambda_2}{d\tau^2} + \cos\lambda_2\left(\frac{d\lambda_2}{d\tau}\right)^2\right] + \quad, \\ \cos\lambda_3 &= \cos\lambda_2 - \tau_1\sin\lambda_2\frac{d\lambda_2}{d\tau} - \tfrac{1}{2}\tau_1^2\left[\sin\lambda_2\frac{d^2\lambda_2}{d\tau^2} + \cos\lambda_2\left(\frac{d\lambda_2}{d\tau}\right)^2\right] + \quad, \end{aligned}$$

and similar expressions in β_1, β_3, Λ_1, and Λ_3 Substituting these and similar expansions in β_1 and β_3 in equations (3) and reducing, it is found that

$$
(10)\begin{cases}
\Delta = \frac{1}{2}\tau_1\tau_2\tau_3 \cos\beta_2\left(\frac{d\beta_2}{d\tau}\frac{d^2\lambda_2}{d\tau^2} - \frac{d^2\beta}{d\tau}\frac{d\lambda_2}{d\tau}\right) - \tau_1\tau\ \tau_3 \sin\beta_2\left(\frac{d\beta_2}{d\tau}\right)^2\frac{d\lambda_2}{d\tau} + \quad, \\
A = R_1\left\{-\tau\ \sin(\lambda_2 - \Lambda_2)\frac{d\beta}{d\tau} + \tau_2 \sin\beta\ \cos\beta\ \cos(\lambda_2 - \Lambda_2)\frac{d\lambda_2}{d\tau} + \quad\right\}, \\
B = R_3\left\{-\tau\ \sin(\lambda_2 - \Lambda_2)\frac{d\beta}{d\tau} + \tau_2 \sin\beta_2 \cos\beta_2 \cos(\lambda_2 - \Lambda\ )\frac{d\lambda_2}{d\tau} + \quad\right\}, \\
C = R_2\left\{-\tau_2 \sin(\lambda\ - \Lambda_2)\frac{d\beta_2}{d\tau} + \tau_2 \sin\beta\ \cos\beta\ \cos(\lambda\ - \Lambda_2)\frac{d\lambda_2}{d\tau} + \quad\right\}
\end{cases}
$$

From these equations it follows that Δ is of the third order while A, B, and C are each of the first order

Both members of equation (8) may be considered as being power series in τ_1, τ_2, τ_3, inasmuch as the left member starts with terms of the third degree the right member must also start with terms of the third degree, since the coefficients of corresponding powers in the two members are necessarily equal That means that the terms of the second and third degrees in P and Q destroy each other The question which arises here is whether both P and Q contain terms of the third degree If P should start with terms of the third degree and Q with terms of the fourth degree or higher, then $\Delta\rho = P$ would give an approximate value of ρ_2, and through this of r_2, after which the higher terms in (8) could be included But if Q contains terms of the third degree, then equation (8) must be treated as involving the two unknowns ρ_2 and r_2

It is easily shown that Q contains terms of the third degree, but considerable work is required to get the explicit expression for P It follows from the fact that β_1 and β_3 appear in A, B, and C in the same way, and from the form of the first of (9), that P does not contain any terms which do not involve derivatives of Λ_2 or R_2 Making use of this fact to shorten the work, it is found after somewhat lengthy, though simple, reductions that

$$
(11)\begin{cases}
P = \frac{1}{2}\tau_1\tau_2\tau_3 \cos(\lambda_2 - \Lambda_2)\frac{d\beta_2}{d\tau}\left\{R_2\frac{d^2\Lambda_2}{d\tau^2} + 2\frac{d\Lambda_2}{d\tau}\frac{dR_2}{d\tau}\right\} \\
\qquad + \frac{1}{2}\tau_1\tau\ \tau_3 \sin(\lambda_2 - \Lambda_2)\frac{d\beta_2}{d\tau}\left\{R_2\left(\frac{d\Lambda_2}{d\tau}\right)^2 - \frac{d\ R_2}{d\tau^2}\right\} \\
\qquad + \frac{1}{2}\tau_1\tau_2\tau_3 \sin\beta_2 \cos\beta_2 \sin(\lambda_2 - \Lambda_2)\frac{d\lambda_2}{d\tau}\left\{R_2\frac{d^2\Lambda_2}{d\tau^2} + 2\frac{d\Lambda_2}{d\tau}\frac{dR_2}{d\tau}\right\} \\
\qquad - \frac{1}{2}\tau_1\tau_2\tau_3 \sin\beta_2 \cos\beta_2 \cos(\lambda_2 - \Lambda_2)\frac{d\lambda_2}{d\tau}\left\{R_2\left(\frac{d\Lambda_2}{d\tau}\right)^2 - \frac{d^2R_2}{d\tau}\right\} \\
\qquad + \quad, \\
Q = -\frac{1}{2}\tau_1\tau\ \tau_3 R\ \left\{\sin(\lambda_2 - \Lambda_2)\frac{d\beta_2}{d\tau} - \sin\beta_2\cos\beta_2\cos(\lambda_2 - \Lambda_2)\frac{d\lambda_2}{d\tau}\right\} +
\end{cases}
$$

From these equations it is seen that P and Q are each of the third order in the τ_i, agreeing with what was previously stated Consequently equation (2) alone does not define ρ_2

A relation exists between ρ_2 and r_2 which, with (8), defines these quantities Consider the triangle formed by the body, the sun, and

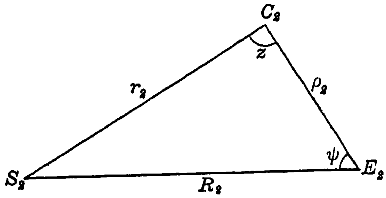

Fig 57

the earth at the epoch t_2 Denote the angle at the earth by ψ and the one at the unknown body by z The following equations result from the relations among the sides and angles of the plane triangle

$$(12)\qquad \begin{cases} \rho_2 = \dfrac{R_2 \sin(\psi + z)}{\sin z}, \\ r_2 = \dfrac{R_2 \sin \psi}{\sin z} \end{cases}$$

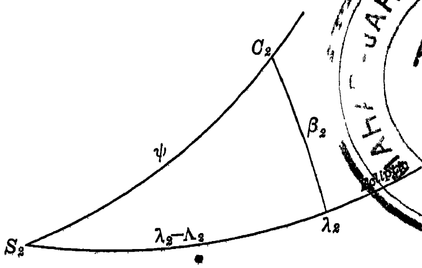

Fig 58

The angle ψ, which must lie between zero and π, is given by the equation

$$(13)\qquad \cos\psi = \cos\beta_2 \cos(\lambda_2 - \Lambda_2)$$

Introducing equations (12) in equation (8), the latter becomes

$$(14)\qquad \Delta R_2 \sin\psi \cos z + (\Delta R_2 \cos\psi - P)\sin z = \frac{Q}{R_2^3 \sin^3\psi} \sin^4 z$$

To simplify this expression let

$$
(15) \qquad \begin{cases} N \sin n = \Delta R_2 \sin \psi, \\ N \cos n = \Delta R \;\; \cos \psi - P, \\ \qquad M = \dfrac{Q}{N R_2^3 \sin^3 \psi} \end{cases}
$$

The sign of N will be taken the same as the sign of Q, from which it follows that M will always be positive Then (14) becomes

$$
(16) \qquad M \sin^4 z = \sin (z + n)
$$

When z has been found from this equation (12) gives ρ and (4), (5), or (6), ρ_1 and ρ_3, from which the elements may be computed

229 Solution of Equation (16) Equation (16) is apparently transcendental, but it may be easily reduced to an algebraic equation of the eighth degree in $\sin z$ Expanding $\sin(z+n)$ and squaring, it is found that

$$
(17) \qquad M^2 \sin^8 z - 2M \cos n \sin^5 z + \sin^2 z - \sin^2 n = 0
$$

It follows from the second equation of (12) and the relations among the angles of the triangle $S_2 C E$ (Fig 57) that z must be real, positive, and less than $\pi - \psi$ in order to have any meaning in the problem

The motion of the earth satisfies the conditions of the problem, viz that the motion shall be in a plane passing through the center of the sun, and that the law of areas shall hold One root of (17) should, therefore, correspond to the orbit of the earth, and it is evident from (12) that it is the real, positive value $z = \pi - \psi$

The question of the possibility of the existence of other real positive roots will now be investigated Suppose $\cos n$ is negative, then there is but one variation in the signs of (17) and hence but one real positive root This is the root $\sin z = \sin(\pi - \psi)$ which is always present, and there is, therefore, no solution to the problem in this case

Suppose $\cos n$ is positive, then it follows from the variations of the signs of (17) that there are not more than three real positive roots One is $\sin z = \sin(\pi - \psi)$ Let the values of z in the other two be denoted by z_1 and z_2 The following three cases are possible

$$
\begin{aligned}
&(a) \quad z_1 \leqq z_2 \leqq \pi - \psi, \\
&(b) \quad z_1 \leqq \pi - \psi \leqq z, \\
&(c) \quad \pi - \psi \leqq z_1 \leqq z
\end{aligned}
$$

In (a) z_1 and z_2 satisfy the conditions imposed But if $z = \pi - \psi$ is a solution of (17) $z = \psi$ is also a solution, because $\sin \psi = \sin(\pi - \psi)$ The second of these has been introduced in the process of rationalization and will not satisfy (16) Either z_1 or z_2 will equal ψ and must be excluded, being a solution foreign to the problem In this manner it is determined which of the two roots of (a) is to be taken In (b) z_1 alone satisfies the conditions of the problem In (c) there is no solution

The roots of (17) may be found by the ordinary methods of locating the roots of algebraic equations Or, if the observed body is a small planet, it will generally be seen near opposition and $\sin \psi$ will be a small quantity It follows from the second equation of (12) that $\sin z$ will also be small Then $\sin^4 z$ will be very small, and $z_0 = -n$ may be taken as an approximate value of z A more nearly correct value may be found by differential corrections Equation (16) may be written

$$M \sin^4(z_0 + \Delta z_0) - \sin(n + z_0 + \Delta z_0) = 0$$

Expanding by Taylor's formula, neglecting terms of order higher than the first in Δz_0 and putting $z_0 = -n$, the value of the correction is found to be given by

$$(18) \qquad \Delta z_0 = \frac{M \sin^4 n}{4M \sin^3 n \cos n + 1},$$

where Δz_0 is expressed in radians The more approximate value of z is $z_1 = z_0 + \Delta z_0$ The process may be repeated until the correct value has been found

When the observed body is not near opposition an approximate value of the solution of (16) may be found by a few trials, and more and more correct values by differential corrections in the manner which has just been explained

When z has been found r_2 and ρ_2 are given by equations (12) and ρ_1 and ρ_3 by equations (4), (5), or (6) In using the ratios of the triangles $[r_i, r_j]$ the terms of the second order, which involve r_2, should be included in order to obtain as great accuracy as possible Thus all of the coordinates at the three times of observation are known The results are, however, subject to some slight uncertainties owing to the approximations employed Before the computation is carried further the results already found should be improved as much as possible The corrections for parallax, time, and aberration should be applied The terms of the first and higher orders in the expansions, if they are sensible, should be included The terms of the first order involve $\frac{dr_2}{d\tau}$ which evidently cannot be accurately computed

until a, e, and T have been found, but it will be sufficiently exact to substitute for the derivative at the time t the average change in the radius vector during the interval $(t_3 - t_1)$ That is

$$\frac{dr_2}{d\tau} = \frac{r_3 - r_1}{k(t_3 - t_1)} + \epsilon, \tag{19}$$

where ϵ is so small that it generally may be neglected The terms of higher order in the ratios of the triangles in (4), (5), and (6) should be included until they become insensible The applications generally will be to the planetoid orbits where the eccentricity is not large and where the higher terms of these developments decrease with great rapidity

Determination of the Elements

230 Determination of the Heliocentric Coordinates, Node, Inclination, and Argument of Latitude The steps are precisely the same as the corresponding ones in the case of parabolic orbits, and it will be sufficient to rewrite the equations

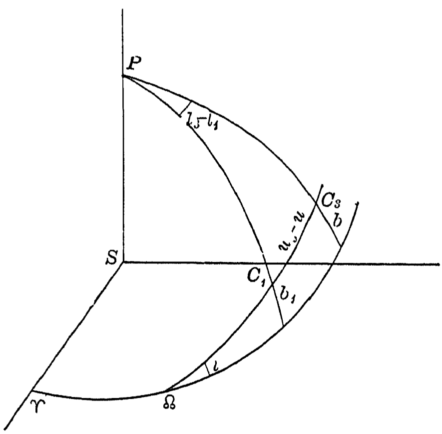

Fig 59

The heliocentric coordinates are given by

$$(20)\quad \begin{cases} r_1 \cos b_1 \cos(l_1 - \Lambda_1) = \rho_1 \cos\beta_1 \cos(\lambda_1 - \Lambda_1) - R_1 \cos B_1, \\ r_1 \cos b_1 \sin(l_1 - \Lambda_1) = \rho_1 \cos\beta_1 \sin(\lambda_1 - \Lambda_1), \\ r_1 \sin b_1 \qquad\qquad\qquad = \rho_1 \sin\beta_1 - R_1 \sin B_1, \\ r_3 \cos b_3 \cos(l_3 - \Lambda_3) = \rho_3 \cos\beta_3 \cos(\lambda_3 - \Lambda_3) - R_3 \cos B_3, \\ r_3 \cos b_3 \sin(l_3 - \Lambda_3) = \rho_3 \cos\beta_3 \sin(\lambda_3 - \Lambda_3), \\ r_3 \sin b_3 \qquad\qquad\qquad = \rho_3 \sin\beta_3 - R_3 \sin B_3 \end{cases}$$

The node and inclination are given by

$$(21)\quad \begin{cases} \tan i \sin (l_1 - \Omega) = \tan b_1, \\ \tan i \cos (l_1 - \Omega) = \dfrac{\tan b_3 - \tan b_1 \cos (l_3 - l_1)}{\sin (l_3 - l_1)}, \quad (0 \lesseqgtr i \lesseqgtr 180^\circ) \end{cases}$$

The arguments of the latitude are given by

$$(22)\quad \begin{cases} \left.\begin{aligned} \tan u_1 &= \dfrac{\tan (l_1 - \Omega)}{\cos i} \\ \tan u_3 &= \dfrac{\tan (l_3 - \Omega)}{\cos i}, \end{aligned}\right\} \text{(If } i < 45^\circ \text{ or } i > 135^\circ \text{)} \\ \left.\begin{aligned} \tan u_1 &= \dfrac{\tan b_1}{\cos (l_1 - \Omega) \sin i}, \\ \tan u_3 &= \dfrac{\tan b_3}{\cos (l_3 - \Omega) \sin i}, \end{aligned}\right\} \text{(If } 45^\circ < i < 135 \text{)} \\ \left.\begin{aligned} \sin u_1 &= \dfrac{\sin b_1}{\sin i}, \\ \sin u_3 &= \dfrac{\sin b_3}{\sin i} \end{aligned}\right\} \text{(Define the quadrant)} \end{cases}$$

Let P represent the pole of the ecliptic, and C_1 and C_3 the projections on the unit sphere of the places of the body at the epochs t_1 and t_3 From the spherical triangle PC_1C_3 it follows that

$$(23)\qquad \cos (u_3 - u_1) = \sin b_1 \sin b_3 + \cos b_1 \cos b_3 \cos (l_3 - l_1),$$

which may be used as a control formula on the work up to this point

231 The Method of Gauss The four elements a, e, ω, and T remain to be found The first three are geometrical and may be obtained independently of the last one, which may be computed from the law of areas when the rest are known

In the case of the parabolic orbit there were but two elements q and ω in place of the three a, e, and ω, which were required, and they were determined by the two points on the parabola. In the case of an elliptic orbit another known quantity must be employed, and it has been found convenient, as Gauss has shown, to use for it the interval of time between the first and third observations In other words, when the focus and two points are given only two parabolas are determined, one of which will be excluded by the second observation. On the other hand an infinite number of ellipses having a common focus may pass through two points But there is, in general, only one of these in which a body will move from the first position to the third, under the action of a given force, in the interval of time $t_3 - t_1$

Instead of computing a, e, and ω directly it is convenient to find first certain auxiliary quantities from which they can be obtained Gauss used for this purpose the ratio of the sector to the triangle contained between the radii at the epochs t_1 and t_3, and one-half of the difference of the eccentric anomalies at these two epochs Two equations must be obtained containing these quantities alone as unknowns

232 The First Equation of Gauss From the definition of $[r_1, r_3]$ it follows that

$$[r_1, r_3] = \tfrac{1}{2} r_1 r_3 \sin(v_3 - v_1) - \tfrac{1}{2} r_1 r_3 \sin(u_3 - u_1) \tag{24}$$

Let A_{13} represent the area of the sector contained between the radii r_1 and r_3 Then it follows from the law of areas that

$$A_{13} = \tfrac{1}{2} k \sqrt{p}\,(t_3 - t_1) = \tfrac{1}{2} \tau \sqrt{p} \tag{25}$$

Let η represent the ratio of the area of the sector to that of the triangle, or

$$\eta = \frac{A_{13}}{[r_1, r_3]} = \frac{\tau \sqrt{p}}{r_1 r_3 \sin(u_3 - u_1)} \tag{26}$$

Since p is unknown it must be eliminated from this equation in order to reduce it to the required type

The polar equation of the conic gives

$$\begin{cases} \dfrac{p}{r_1} = 1 + e\cos v_1, \\[2ex] \dfrac{p}{r_3} = 1 + e\cos v_3, \end{cases}$$

whence

$$p\,\frac{r_1 + r_3}{r_1 r_3} = 2 + e(\cos v_1 + \cos v_3) = 2 + 2e\cos\left(\frac{v_3 + v_1}{2}\right)\cos\left(\frac{v_3 - v_1}{2}\right), \tag{27}$$

$v_3 - v_1 = u_3 - u_1$ is known, the only unknown in the right member being $e\cos\left(\dfrac{v_3 + v_1}{2}\right)$, which will now be eliminated From the equations of Art 98 it follows that

$$\begin{cases} \sqrt{r_1}\cos\dfrac{v_1}{2} = \sqrt{a(1-e)}\cos\dfrac{E_1}{2}, \\[2ex] \sqrt{r_1}\sin\dfrac{v_1}{2} = \sqrt{a(1+e)}\sin\dfrac{E_1}{2}, \\[2ex] \sqrt{r_3}\cos\dfrac{v_3}{2} = \sqrt{a(1-e)}\cos\dfrac{E_3}{2}, \\[2ex] \sqrt{r_3}\sin\dfrac{v_3}{2} = \sqrt{a(1+e)}\sin\dfrac{E_3}{2} \end{cases}$$

From these equations it is found that

$$(28)\quad \begin{cases} \sqrt{r_1 r_3}\cos\left(\dfrac{v_3-v_1}{2}\right) = a\cos\left(\dfrac{E_3-E_1}{2}\right) - ae\cos\left(\dfrac{E_3+E_1}{2}\right), \\ \sqrt{r_1 r_3}\cos\left(\dfrac{v_3+v_1}{2}\right) = a\cos\left(\dfrac{E_3+E_1}{2}\right) - ae\cos\left(\dfrac{E_3-E_1}{2}\right) \end{cases}$$

Eliminating $e\cos\left(\dfrac{E_3+E_1}{2}\right)$ and solving for $e\cos\left(\dfrac{v_3+v_1}{2}\right)$, it is found that

$$(29)\qquad e\cos\left(\frac{v_3+v_1}{2}\right) = \frac{p}{\sqrt{r_1 r_3}}\cos\left(\frac{E_3-E_1}{2}\right) - \cos\left(\frac{v_3-v_1}{2}\right)$$

As a consequence of this equation (27) reduces to

$$p = \frac{2r_1 r_3 \sin^2\left(\dfrac{v_3-v_1}{2}\right)}{r_1+r_3-2\sqrt{r_1 r_3}\cos\left(\dfrac{v_3-v_1}{2}\right)\cos\left(\dfrac{E_3-E_1}{2}\right)}$$

Eliminating p from this equation and (26) the equation

$$(30)\quad \eta^2 = \frac{\tau_2^2}{2r_1 r_3\cos^2\left(\dfrac{v_3-v_1}{2}\right)\left\{r_1+r_3-2\sqrt{r_1 r_3}\cos\left(\dfrac{v_3-v_1}{2}\right)\cos\left(\dfrac{E_3-E_1}{2}\right)\right\}}$$

is obtained In order to simplify it let

$$(31)\quad \begin{cases} v_3-v_1 = u_3-u_1 = 2f, \\ E_3-E_1 = 2g, \\ m = \dfrac{\tau_2}{(2\sqrt{r_1 r_3}\cos f)^{\frac{3}{2}}}, \\ l = \dfrac{r_1+r_3}{4\sqrt{r_1 r_3}\cos f} - \dfrac{1}{2} \end{cases}$$

Then (30) reduces to

$$(32)\qquad \eta^2 = \frac{m^2}{l+\sin^2\dfrac{g}{2}},$$

in which η and g are the unknowns This is the first equation in the method of Gauss

233 The Second Equation of Gauss. An independent equation involving η and g will now be derived. It will be made to depend upon Kepler's equation, thus insuring its independence of (32)

which was derived without reference to Kepler's equation The first equations are

$$\begin{cases} M_1 = \dfrac{k(t_1 - T)}{a^{\frac{3}{2}}} = E_1 - e \sin E_1, \\ M_3 = \dfrac{k(t_3 - T)}{a^{\frac{3}{2}}} = E_3 - e \sin E_3, \end{cases}$$

whence

$$(33) \qquad \frac{k(t_3 - t_1)}{a^{\frac{3}{2}}} = \frac{\tau_2}{a^{\frac{3}{2}}} = 2g - 2e \sin g \cos\left(\frac{E_3 + E_1}{2}\right)$$

The quantities a and $e \cos\left(\frac{E_3 + E_1}{2}\right)$ must be eliminated in order to reduce this equation to the required type By means of the first equation of (28), equation (33) becomes

$$(34) \qquad \frac{\tau_2}{a^{\frac{3}{2}}} = 2g - \sin 2g + 2\frac{\sqrt{r_1 r_3}}{a} \sin g \cos f$$

It remains to eliminate a By Art 98

$$\begin{cases} \dfrac{r_1}{a} = 1 - e \cos E_1, \\ \dfrac{r_3}{a} = 1 - e \cos E_3, \end{cases}$$

whence

$$\frac{r_1 + r_3}{a} = 2 - 2e \cos g \cos\left(\frac{E_3 + E_1}{2}\right)$$

Eliminating $e \cos\left(\frac{E_3 + E_1}{2}\right)$ by the first equation of (28) this equation becomes

$$\frac{1}{a} = \frac{2 \sin g}{r_1 + r_3 - 2\sqrt{r_1 r_3} \cos g \cos f},$$

which becomes as a consequence of (30)

$$(35) \qquad \frac{1}{a} = \left[\frac{2\eta \sin g \cos f}{\tau}\right]^2 r_1 r_3$$

Eliminating a between (34) and (35) it is found that

$$(36) \qquad \frac{\eta^3}{m} - \frac{\eta^2}{m^2} = \frac{2g - \sin 2g}{\sin^3 g},$$

which is the second equation in η and g

234 Solution of (32) and (36) It follows from (26) that η is positive if the heliocentric motion in the orbit is less than 180° in the interval $t_3 - t_1$ It will be supposed in what follows that the observations are so close together that this condition is fulfilled

Let

$$(37)\qquad \begin{cases} \sin^2 \dfrac{g}{2} = x, \\ \dfrac{2g - \sin 2g}{\sin^3 g} = X \end{cases}$$

Eliminating η from (32) and (36) and making use of (37), it is found that

$$(38)\qquad m = (l + x)^{\frac{1}{2}} + X(l + x)^{\frac{3}{2}}$$

X must now be expressed in terms of x, when (38) will involve this quantity alone as an unknown This will be done by first expressing X in terms of g, and then g in terms of x The following are well-known expansions of the trigonometrical functions

$$\sin 2g = 2g - \tfrac{4}{3} g^3 + \tfrac{4}{15} g^5 - \tfrac{8}{315} g^7 + \tfrac{8}{5670} g^9 - \quad ,$$
$$\sin^3 g = g^3 - \tfrac{1}{2} g^5 + \tfrac{13}{120} g^7 - \tfrac{41}{3024} g^9 + \quad ,$$

whence

$$(39)\qquad X = \frac{\tfrac{4}{3} - \tfrac{4}{15} g^2 + \tfrac{8}{315} g^4 - \tfrac{8}{5670} g^6 +}{1 - \tfrac{1}{2} g^2 + \tfrac{13}{120} g^4 - \tfrac{41}{3024} g^6 +}$$
$$= \tfrac{4}{3}\left(1 + \tfrac{3}{10} g^2 + \tfrac{17}{280} g^4 + \tfrac{29}{2800} g^6 + \quad \right)$$

From the first of (37) it follows that

$$g = 2 \sin^{-1}(x^{\frac{1}{2}}) = 2x^{\frac{1}{2}} + \tfrac{1}{3} x^{\frac{3}{2}} + \tfrac{3}{20} x^{\frac{5}{2}} + \tfrac{5}{56} x^{\frac{7}{2}} + \quad ,$$
$$g^2 = 4x + \tfrac{4}{3} x^2 + \tfrac{32}{45} x^3 + \tfrac{16}{35} x^4 + \quad ,$$
$$g^4 = 16x^2 + \tfrac{32}{3} x^3 + \tfrac{112}{15} x^4 + \quad ,$$
$$g^6 = 64x^3 + 64x^4 +$$

Then (39) becomes

$$(40)\qquad X = \frac{4}{3}\left(1 + \frac{6}{5} x + \frac{6\ 8}{5\ 7} x^2 + \frac{6\ 8\ 10}{5\ 7\ 9} x^3 + \quad \right),$$

or

$$(41)\qquad X = \frac{1}{\dfrac{3}{4}\left[1 + \dfrac{6}{5} x + \dfrac{6\ 8}{5\ 7} x^2 + \dfrac{6\ 8\ 10}{5\ 7\ 9} x^3 + \quad \right]^{-1}}$$
$$= \frac{1}{\dfrac{3}{4} - \dfrac{9}{10}\left(x - \dfrac{2}{35} x^2 - \dfrac{52}{1575} x^3 \quad \right)}$$

Let

$$(42)\qquad \xi = \frac{2}{35} x^2 + \frac{52}{1575} x^3 +$$

If $\frac{1}{2} g$ is a small quantity of the first order, x is of the second order and ξ is of the fourth order

From (32) it is found that

$$x = \frac{m^2}{\eta^2} - l \tag{43}$$

Let

$$h = \frac{m^2}{\frac{5}{6} + l + \xi}, \tag{44}$$

then (36) may be written

$$\eta - 1 = \frac{m^2 X}{\eta} = \frac{\frac{10}{9} h}{\eta^2 - h},$$

from which it is found that

$$\eta^3 - \eta^2 - h\eta - \frac{h}{9} = 0 \tag{45}$$

If ξ were known h would be determined by (44) and η by (45), which has but one real positive root In the first approximation compute h assuming that the small quantity ξ is zero then find the real positive root of (45) Or, instead of computing the root, use may be made of the tables which have been constructed by Gauss, giving the real positive values of η for values of h ranging from 0 to 0 6* The value of x is then computed by (43) and the value of ξ by (42)† With this value of ξ, h and η are recomputed, and the process is repeated until the desired degree of precision is attained, when the value of g will be given by the first of (37)

The species of conic section is decided at this point, the orbit being an ellipse, parabola, or hyperbola according as x is positive, zero, or negative, for, $x = \sin^2 \frac{g}{2} = \sin \frac{1}{4}(E_3 - E_1)$, and E_3 and E_1 are real in ellipses, zero in parabolas, and imaginary in hyperbolas

Gauss has introduced a transformation which facilitates the computation of l which was defined in the last equation of (31)‡ Let

$$\sqrt[4]{\frac{r_3}{r_1}} = \tan(45 + \omega'), \quad 0 \leqq \omega' \leqq 45^\circ, \tag{46}$$

whence

$$\frac{r_1 + r_3}{\sqrt{r_1 r_3}} = \sqrt{\frac{r_3}{r_1}} + \sqrt{\frac{r_1}{r_3}} = \tan^2(45^\circ + \omega') + \cot^2(45 + \omega),$$

* This table is XIII in Watson's *Theoretical Astronomy*, and VIII in Oppolzer's *Bahnbestimmung*

† The value of ξ with argument x is given in Watson's *Theoretical Astronomy*, Table XIV, and in Oppolzer's *Bahnbestimmung*, Table IX

‡ *Theoria Motus*, Art 86

or

$$\frac{r_1 + r_3}{\sqrt{r_1 r_3}} = 2 + 4 \tan^2 2\omega'$$

Then the last equation of (31) becomes

$$l = \frac{\sin^2 \frac{f}{2} + \tan^2 2\omega'}{\cos f} \tag{47}$$

235 Computation of the Elements Let $e = \sin \phi$ and it follows that

$$\begin{cases} \sqrt{1 + e} = \cos \frac{\phi}{2} + \sin \frac{\phi}{2}, \\ \sqrt{1 - e} = \cos \frac{\phi}{2} - \sin \frac{\phi}{2} \end{cases}$$

Then, if the orbit is an ellipse, the following equations are true (Art 98)

$$\begin{cases} \sqrt{r_1} \sin \frac{v_1}{2} = \sqrt{a} \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right) \sin \frac{E_1}{2}, \\ \sqrt{r_1} \cos \frac{v_1}{2} = \sqrt{a} \left(\cos \frac{\phi}{2} - \sin \frac{\phi}{2} \right) \cos \frac{E_1}{2}, \\ \sqrt{r_3} \sin \frac{v_3}{2} = \sqrt{a} \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right) \sin \frac{E_3}{2}, \\ \sqrt{r_3} \cos \frac{v_3}{2} = \sqrt{a} \left(\cos \frac{\phi}{2} - \sin \frac{\phi}{2} \right) \cos \frac{E_3}{2} \end{cases} \tag{48}$$

Let

$$v_3 + v_1 = 2F, \qquad E_3 + E_1 = 2G$$

Then, making use of equations (31), equations (48) may be written

$$\begin{cases} \sqrt{\frac{r_1}{a}} \sin \left(\frac{F - f}{2} \right) = \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right) \sin \left(\frac{G - g}{2} \right), \\ \sqrt{\frac{r_1}{a}} \cos \left(\frac{F - f}{2} \right) = \left(\cos \frac{\phi}{2} - \sin \frac{\phi}{2} \right) \cos \left(\frac{G - g}{2} \right), \\ \sqrt{\frac{r_3}{a}} \sin \left(\frac{F + f}{2} \right) = \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2} \right) \sin \left(\frac{G + g}{2} \right), \\ \sqrt{\frac{r_3}{a}} \cos \left(\frac{F + f}{2} \right) = \left(\cos \frac{\phi}{2} - \sin \frac{\phi}{2} \right) \cos \left(\frac{G + g}{2} \right), \end{cases} \tag{49}$$

in which F, G, a, and ϕ are the unknowns

Now

$$
(50)\quad \begin{cases}
\sqrt{\dfrac{r_3}{a}} - \sqrt{\dfrac{r_1}{a}} = \sqrt[4]{\dfrac{r_1 r_3}{a}}\left(\sqrt[4]{\dfrac{r_3}{r_1}} - \sqrt[4]{\dfrac{r_1}{r_3}}\right) \\[2ex]
\qquad = \sqrt[4]{\dfrac{r_1 r_3}{a}}\left\{\tan(45^\circ + \omega') - \cot(45^\circ + \omega')\right\} \\[2ex]
\qquad = 2\sqrt[4]{\dfrac{r_1 r_3}{a}}\tan 2\omega'
\end{cases}
$$

Multiply equations (49) respectively by the factors in each of the following four columns

$+\sin\left(\dfrac{F+g}{2}\right)$	$+\cos\left(\dfrac{F+g}{2}\right)$	$+\sin\left(\dfrac{F-g}{2}\right)$	$+\cos\left(\dfrac{F-g}{2}\right)$
$+\cos\left(\dfrac{F+g}{2}\right)$	$-\sin\left(\dfrac{F+g}{2}\right)$	$+\cos\left(\dfrac{F-g}{2}\right)$	$-\sin\left(\dfrac{F-g}{2}\right)$
$-\sin\left(\dfrac{F-g}{2}\right)$	$-\cos\left(\dfrac{F-g}{2}\right)$	$-\sin\left(\dfrac{F+g}{2}\right)$	$-\cos\left(\dfrac{F+g}{2}\right)$
$-\cos\left(\dfrac{F-g}{2}\right)$	$+\sin\left(\dfrac{F-g}{2}\right)$	$-\cos\left(\dfrac{F+g}{2}\right)$	$+\sin\left(\dfrac{F+g}{2}\right)$

Adding the four sets of products and making use of (50), it follows that

$$
(51)\quad \begin{cases}
\left(\sqrt[4]{\dfrac{a}{r_1 r_3}}\sin g\right)\cos\dfrac{\phi}{2}\sin\left(\dfrac{F-G}{2}\right) = \cos\left(\dfrac{f+g}{2}\right)\tan 2\omega', \\[2ex]
\left(\sqrt[4]{\dfrac{a}{r_1 r_3}}\sin g\right)\cos\dfrac{\phi}{2}\cos\left(\dfrac{F-G}{2}\right) = \sin\left(\dfrac{f+g}{2}\right)\sec 2\omega', \\[2ex]
\left(\sqrt[4]{\dfrac{a}{r_1 r_3}}\sin g\right)\sin\dfrac{\phi}{2}\sin\left(\dfrac{F+G}{2}\right) = \cos\left(\dfrac{f-g}{2}\right)\tan 2\omega', \\[2ex]
\left(\sqrt[4]{\dfrac{a}{r_1 r_3}}\sin g\right)\sin\dfrac{\phi}{2}\cos\left(\dfrac{F+G}{2}\right) = \sin\left(\dfrac{f-g}{2}\right)\sec 2\omega'
\end{cases}
$$

The right members of these equations are entirely known, therefore they determine ϕ (which lies between 0 and 90°), a (which is positive), $F-G$, and $F+G$ uniquely. Then

$$
(52)\quad \begin{cases}
a = a, \\
e = \sin\phi, \\
E_1 = G - g, \\
E_3 = G + g, \\
v_1 = F - f, \\
v_3 = F + f, \\
\omega = u_3 - v_3 = u_1 - v_1
\end{cases}
$$

The following equations may be used as check formulas

$$(53)\quad \begin{cases} \tan\dfrac{v_1}{2} = \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E_1}{2} = \tan\left(45 + \dfrac{\phi}{2}\right)\tan\dfrac{E_1}{2}, \\ \tan\dfrac{v_3}{2} = \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E_3}{2} = \tan\left(45^\circ + \dfrac{\phi}{2}\right)\tan\dfrac{E_3}{2}, \\ r_1 = a\,(1 - \sin\phi\cos E_1), \\ r_3 = a\,(1 - \sin\phi\cos E_3) \end{cases}$$

The one remaining element, the time of perihelion passage, is found as follows

$$\frac{k\,(t_1 - T)}{a^{\frac{3}{2}}} = M_1 = E_1 - e\sin E_1,$$
$$\frac{k\,(t_3 - T)}{a^{\frac{3}{2}}} = M_3 = E_3 - e\sin E_3,$$

whence

$$(54)\qquad T = t_1 - \frac{a^{\frac{3}{2}}}{k}(E_1 - e\sin E_1) = t_3 - \frac{a^{\frac{3}{2}}}{k}(E_3 - e\sin E_3)$$

The mean motion, n, is given by

$$n = \frac{k}{a^{\frac{3}{2}}}$$

Instead of the time of perihelion passage the mean anomaly at the epoch is frequently used as the last element If T_0 represents the epoch, the mean anomaly at the epoch is given by

$$(55)\qquad M_0 = M_1 + n\,(T_0 - t_1) = M_3 + n\,(T_0 - t_3)$$

This completes the method of Gauss for the determination of the elements of elliptic orbits

236 Second Method of Determining the Elements* The method of Gauss is quite complicated theoretically and is rather long in application Moreover, it requires the assumption that the heliocentric motion is less or greater than 180 (which defines the sign of η), and it differs for the three classes of conics The question arises whether these faults may not be avoided

In the method of Gauss r_2 and ρ_2 were used only in finding r_1, r_3, ρ_1, ρ_3, in the following they will play a more important part. Suppose equations with subscripts 2 are added to (20) and (22), and that r_1, r_2, r_3, u_1, u , u_3 have been computed as in the method which has been

* Read before the American Assoc Adv Science by F R Moulton, Aug 1901, and published in the *Astronomical Journal*, No 510

explained The elements i and Ω may be computed by equations (21), which are valid for any orbit The difficulties all arose in finding a, e, ω Let the element p be adopted in place of a It is more convenient in that it does not become infinite when e equals unity, and it is involved alone in the equation of areas,

$$k\sqrt{p}\,dt = r^2 dv = r\ du$$

The integral of this equation is

$$(56) \qquad k\sqrt{p}\,(t_3 - t_1) = \int_{u_1}^{u_3} r^2\, du$$

If r^2 were expressed in terms of u the integral in the right member could be found, when the value of p would be given It will be shown that from the knowledge of the value r^2 when $u = u_1, u_2, u_3$, viz $r^2 = r_1^2$, r_2^2, r_3^2, it may be expressed in terms of u with sufficient accuracy to give a very close approximation to the value of p

For values of u not too remote from u_2 r may be expanded in a converging series of the form

$$(57) \quad r^2 = r_2^2 + c_1(u - u_2) + c_2(u - u_2)^2 + c_3(u - u_2)^3 + c_4(u - u_2)^4 + \cdots,$$

where

$$(58) \quad \begin{cases} c_1 = \dfrac{\partial(r^2)}{\partial u}, & c_2 = \dfrac{1}{2}\dfrac{\partial^2(r^2)}{\partial u_2^2}, & c_3 = \dfrac{1}{6}\dfrac{\partial^3(r^2)}{\partial u_2^3}, \\[2ex] c_4 = \dfrac{1}{24}\dfrac{\partial^4(r^2)}{\partial u_2^4}, & c_5 = \dfrac{1}{120}\dfrac{\partial^5(r^2)}{\partial u_2^5}, & c_6 = \dfrac{1}{720}\dfrac{\partial^6(r^2)}{\partial u_2^6}, \end{cases}$$

In an unknown orbit the coefficients of the series (57) are unknown, but it will now be shown how a sufficient number to define p with the desired degree of accuracy may be easily found By hypothesis, the radii and arguments of latitude are known at the epochs t_1, t_2, and t_3 Hence (57) becomes at t_1 and t_3

$$(59) \quad \begin{cases} r_1^2 = r_2^2 + c_1(u_1 - u_2) + c_2(u_1 - u_2)^2 + c_3(u_1 - u_2)^3 + c_4(u_1 - u_2)^4 + \cdots, \\ r_3^2 = r_2^2 + c_1(u_3 - u_2) + c_2(u_3 - u_2)^2 + c_3(u_3 - u_2)^3 + c_4(u_3 - u_2)^4 + \cdots \end{cases}$$

For abbreviation let

$$(60) \quad \begin{cases} \sigma_1 = u_3 - u_2, \\ \sigma_2 = u_3 - u_1, \\ \sigma_3 = u_2 - u_1, \\ \epsilon_1 = c_3(u_1 - u_2)^3 + c_4(u_1 - u_2)^4 + \cdots = -c_3\sigma_3^3 + c_4\sigma_3^4 + \cdots, \\ \epsilon_3 = c_3(u_3 - u_2)^3 + c_4(u_3 - u_2)^4 + \cdots = +c_3\sigma_1^3 + c_4\sigma_1^4 + \cdots \end{cases}$$

Then equations (59) may be written

$$\begin{cases} -c_1\sigma_3 + c_2\sigma_3^2 = r_1^2 - r_2^2 - \epsilon_1, \\ +c_1\sigma_1 + c_2\sigma_1^2 = r_3^2 - r_2^2 - \epsilon_3 \end{cases}$$

Solving for c_1 and c_3, it is found that

$$\left\{\begin{aligned} c_1 &= \frac{-(r_1^2-\epsilon_1)\,\sigma_1^2+(r_3^2-\epsilon_3)\,\sigma_3^2-r_2^2\,(\sigma_3^2-\sigma_1^2)}{\sigma_1\sigma_2\sigma_3}, \\ c_2 &= \frac{(r_1^2-\epsilon_1)\,\sigma_1+(r_3^2-\epsilon_3)\,\sigma_3-r_2^2\sigma_2}{\sigma_1\sigma_2\sigma_3}, \end{aligned}\right.$$

or, substituting the values of ϵ_1 and ϵ_3,

$$(61)\quad \left\{\begin{aligned} c_1 &= -\frac{r_1^2\sigma_1+r_3^2\sigma_3-r_2^2\,(\sigma_3^2-\sigma_1^2)}{\sigma_1\sigma_2\sigma_3}-c_3\sigma_1\sigma_3-c_4\sigma_1\sigma_3\,(\sigma_1-\sigma_3), \\ c_2 &= \frac{r_1^2\sigma_1+r_3^2\sigma_3-r_2^2\sigma_2}{\sigma_1\sigma_2\sigma_3}-c_3\,(\sigma_1-\sigma_3)+c_4\,(3\sigma_1\sigma_3-\sigma_2^2) \end{aligned}\right.$$

Having obtained these expressions for the coefficients of the second and third terms of (57), let this series be substituted for r^2 in (56) and the integration performed Making use of (60), it is easily found that

$$(62)\quad \left\{\begin{aligned} k\sqrt{p}\,(t_3-t_1) &= r_2^2\sigma_2+\frac{c_1}{2}(\sigma_1^2-\sigma_3^2)+\frac{c_2}{3}(\sigma_1^3+\sigma_3^3)+\frac{c_3}{4}(\sigma_1^4-\sigma_3^4) \\ &\quad +\frac{c_4}{5}(\sigma_1^5+\sigma_3^5)+\cdots \end{aligned}\right.$$

Substituting the values of c_1 and c_2 given in (61), this equation becomes

$$(63)\quad \left\{\begin{aligned} k\sqrt{p}\,(t_3-t_1) &= \frac{r_2^2\sigma_2^3}{6\sigma_1\sigma_3}+\frac{r_1^2\sigma}{6\sigma_3}(2\sigma_3-\sigma_1)+\frac{r_3^2\sigma_2}{6\sigma_1}(2\sigma_1-\sigma_3) \\ &\quad +\frac{c_3\sigma_2^3}{12}(\sigma_3-\sigma_1)-\frac{c_4\sigma_2^3}{30}\{4\,(\sigma_3-\sigma_1)^2+\sigma_1\sigma_3\}-\cdots \end{aligned}\right.$$

If the second observation divides the whole interval into two nearly equal parts, as generally will be the case in practice, σ_1 and σ_3 will be nearly equal Let

$$\sigma_1-\sigma_3=\epsilon, \text{ and } \sigma_1+\sigma_3=\sigma_2,$$

whence

$$\left\{\begin{aligned} \sigma_1 &= \frac{\sigma_2+\epsilon}{2}, \\ \sigma_3 &= \frac{\sigma_2-\epsilon}{2} \end{aligned}\right.$$

Substituting these expressions in the last terms of (63) this equation becomes

$$(64)\quad \left\{\begin{aligned} k\sqrt{p}\,(t_3-t_1) &= \frac{r_2^2\sigma_2^3}{6\sigma_1\sigma_3}+\frac{r_1^2\sigma_2}{6\sigma_3}(2\sigma_3-\sigma_1)+\frac{r_3^2\sigma_2}{6\sigma_1}(2\sigma_1-\sigma_3) \\ &\quad -\frac{c_3\sigma_2^3\epsilon}{12}-\frac{c_4}{120}\sigma_2^3\,(\sigma_2^2+15\epsilon^2)-\cdots \end{aligned}\right.$$

For the intervals of time which are used in determining an orbit this series converges very rapidly*, and an approximate value of p, which is generally as accurate as is desired, may be obtained by taking only the first three terms in the right member of (64). Suppose the value of p has been computed, it will be shown how ω and e may be found.

The polar equation of the conic gives

$$(65)\qquad \begin{cases} e\cos(u_1-\omega)=\dfrac{p-r_1}{r_1}, \\[2mm] e\cos(u_3-\omega)=\dfrac{p-r_3}{r_3}. \end{cases}$$

Now $u_3-\omega=(u_3-u_1)+(u_1-\omega)$. Substituting in the second equation of (65), expanding, and reducing by the first, it is found that

$$(66)\qquad \begin{cases} e\sin(u_1-\omega)=\dfrac{1}{\sin(u_3-u_1)}\left\{\dfrac{p-r_1}{r_1}\cos(u_3-u_1)-\dfrac{p-r_3}{r_3}\right\}, \\[2mm] e\cos(u_1-\omega)=\dfrac{p-r_1}{r_1}. \end{cases}$$

Since e is positive these equations define e and ω uniquely. The value of T is to be found as in the Gaussian method.

If the elements have not been found with sufficient approximation it is now possible to correct them. From the equation

$$r=\frac{p^2}{[1+e\cos(u-\omega)]^2}$$

and equations (58), it is found that

$$(67)\qquad \begin{cases} \dfrac{c_3}{p^2}=\dfrac{-e\sin(u-\omega)}{3[1+e\cos(u-\omega)]^3}+\dfrac{3e\sin(u-\omega)\cos(u-\omega)}{[1+e\cos(u-\omega)]^4}+\dfrac{4e^3\sin^3(u-\omega)}{[1+e\cos(u-\omega)]^5}, \\[2mm] \dfrac{c_4}{p^2}=\dfrac{-e\cos(u-\omega)}{12[1+e\cos(u-\omega)]^3}-\dfrac{e\sin^2(u-\omega)}{[1+e\cos(u-\omega)]^4}+\dfrac{3e\cos(u-\omega)}{4[1+e\cos(u-\omega)]^4} \\[2mm] \qquad\qquad +\dfrac{6e^3\sin^2(u-\omega)\cos(u-\omega)}{[1+e\cos(u-\omega)]^5}+\dfrac{5e^4\sin^4(u-\omega)}{[1+e\cos(u-\omega)]^6}. \end{cases}$$

With the values of c_3 and c_4 computed from these equations the higher terms of (64) may be added, thus determining a more accurate value of p, after which e and ω should be recomputed by (66). Besides being very brief this method has the advantage of being the same for all conics.

* For conditions and rapidity of convergence see original paper in the *Astronomical Journal*, No. 510. It is shown there that the elements of asteroid orbits will be given by the first three terms of (64) correct to the sixth decimal place if the whole interval covered by the observations is not more than 40 days, and in the case of comets' orbits, if the interval is not more than 10 days. When the two corrective terms defined by (67) are added the corresponding intervals are 100 days and 20 days.

IMPROVEMENT OF THE ELEMENTS

237 Variation of two Geocentric Distances In the case of the general conic section there are six elements, and consequently the six coordinates which define two positions of the body have no relation among themselves It follows that the two geocentric distances may be varied independently in this case, and the elements changed so that the second observation and the theory will be in accord But all of the changes of the elements such that the first and third observations shall be fulfilled can be made by varying the geocentric distances ρ_1 and ρ_3

Let the approximate values of ρ_1 and ρ_3 which were used in the derivation of the elements be denoted by $\rho_1^{(0)}$ and $\rho_3^{(0)}$ Let the coordinates computed from the approximate elements at the time t_2 be $\lambda_2^{(0)}$ and $\beta_2^{(0)}$ Let the correct values of the geocentric distances be

$$(68)\qquad \begin{cases} \rho_1 = \rho_1^{(0)} + \Delta\rho_1, \\ \rho_3 = \rho_3^{(0)} + \Delta\rho_3, \end{cases}$$

where $\Delta\rho_1$ and $\Delta\rho_3$ are the corrections to be found

The coordinates of the body at t_2 are functions of ρ_1 and ρ_3 alone as variables, which may be expressed by

$$(69)\qquad \begin{cases} \lambda_2 = \phi\,(\rho_1^{(0)} + \Delta\rho_1,\ \rho_3^{(0)} + \Delta\rho_3), \\ \beta_2 = \psi\,(\rho_1^{(0)} + \Delta\rho_1,\ \rho_3^{(0)} + \Delta\rho_3) \end{cases}$$

Expanding the right members of (69) by Taylor's formula, and neglecting terms of order higher than the first in $\Delta\rho_1$ and $\Delta\rho_3$, it follows that

$$\begin{cases} \lambda_2 = \phi\,(\rho_1^{(0)},\ \rho_3^{(0)}) + \dfrac{\partial\phi}{\partial\rho_1}\Delta\rho_1 + \dfrac{\partial\phi}{\partial\rho_3}\Delta\rho_3, \\[2ex] \beta_2 = \psi\,(\rho_1^{(0)},\ \rho_3^{(0)}) + \dfrac{\partial\psi}{\partial\rho_1}\Delta\rho_1 + \dfrac{\partial\psi}{\partial\rho_3}\Delta\rho_3 \end{cases}$$

But $\phi\,(\rho_1^{(0)},\ \rho_3^{(0)}) = \lambda_2^{(0)}$, $\psi\,(\rho_1^{(0)},\ \rho_3^{(0)}) = \beta_2^{(0)}$, therefore

$$(70)\qquad \Delta\rho_1 = \frac{\begin{vmatrix} \lambda_2 - \lambda_2^{(0)}, & \dfrac{\partial\phi}{\partial\rho_3} \\[2ex] \beta_2 - \beta_2^{(0)}, & \dfrac{\partial\psi}{\partial\rho_3} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial\phi}{\partial\rho_1}, & \dfrac{\partial\phi}{\partial\rho_3} \\[2ex] \dfrac{\partial\psi}{\partial\rho_1}, & \dfrac{\partial\psi}{\partial\rho_3} \end{vmatrix}}, \qquad \Delta\rho_3 = \frac{\begin{vmatrix} \dfrac{\partial\phi}{\partial\rho_1}, & \lambda_2 - \lambda_2^{(0)} \\[2ex] \dfrac{\partial\psi}{\partial\rho_1}, & \beta_2 - \beta_2^{(0)} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial\phi}{\partial\rho_1}, & \dfrac{\partial\phi}{\partial\rho_3} \\[2ex] \dfrac{\partial\psi}{\partial\rho_1}, & \dfrac{\partial\psi}{\partial\rho_3} \end{vmatrix}}$$

These equations determine $\Delta\rho_1$ and $\Delta\rho_3$ when the partial derivatives $\frac{\partial\phi}{\partial\rho_1}$, $\frac{\partial\phi}{\partial\rho_3}$, $\frac{\partial\psi}{\partial\rho_1}$, and $\frac{\partial\psi}{\partial\rho_3}$ have been computed They might be found directly, for the way in which λ and β depend upon the elements is known, but it is more convenient in practice to employ the indirect method explained in the last chapter for effecting similar corrections

Take an arbitrary small variation $\Delta'\rho_1$, and recompute the elements and the theoretical middle position Let the computed coordinates be $\lambda^{(1)}$ and $\beta_2^{(1)}$ Then

$$\left\{\begin{aligned} \lambda_2^{(1)} &= \phi\,(\rho_1^{(0)} + \Delta'\rho_1,\ \rho_3^{(0)}) = \phi\,(\rho_1^{(0)},\ \rho_3^{(0)}) + \frac{\partial\phi}{\partial\rho_1}\Delta'\rho_1 + \quad, \\ \beta_2^{(1)} &= \psi\,(\rho_1^{(0)} + \Delta'\rho_1,\ \rho_3^{(0)}) = \psi\,(\rho_1^{(0)},\ \rho_3^{(0)}) + \frac{\partial\psi}{\partial\rho_1}\Delta'\rho_1 + \end{aligned}\right.$$

Neglecting all except the linear terms, these equations give

$$(71) \qquad \frac{\partial\phi}{\partial\rho_1} = \frac{\lambda^{(1)} - \lambda_2^{(0)}}{\Delta'\rho_1}, \quad \frac{\partial\psi}{\partial\rho_1} = \frac{\beta^{(1)} - \beta^{(0)}}{\Delta'\rho_1},$$

in which the right members are entirely known In a similar manner giving $\rho_3^{(0)}$ an arbitrary increment $\frac{\partial\phi}{\partial\rho_3}$ and $\frac{\partial\psi}{\partial\rho_3}$ may be found Substituting these values of the partial derivatives in (70) the corrections to the geocentric distances are found, after which the improved elements are to be computed as in the first case

This method may be employed on any third observation as well as upon the middle one, and in general the more distant it is from the ones used in finding the preliminary orbit, the more critical is the test and the better are the corrections If it should happen that the corrections are large the higher terms in the expansions, which have been neglected, might become sensible, and the process of correction should be repeated In this manner such a set of elements will be derived that the three theoretical places will agree with the observed

238 Variation of the Elements Suppose there are many observations of the body instead of only three, since they are subject to certain small errors they cannot all be exactly satisfied by the same set of elements The problem now becomes to find the set of elements that will satisfy the observations as a whole in the best possible manner, even though it does not satisfy any particular observation exactly

For abbreviation, let $\alpha_1, \quad , \alpha_6$ represent the six elements of the elliptic orbit Let $\alpha_1^{(0)}, \quad , \alpha_6^{(0)}$ represent the approximate values found

from the three observations Suppose there are n observations made at the epochs $t_1, \ldots, t_n$, and let $\lambda_1, \ldots, \lambda_n$ and $\beta_1, \ldots, \beta_n$ be the observed coördinates Let $\lambda_1^{(0)}, \ldots, \lambda_n^{(0)}$ and $\beta_1^{(0)}, \ldots, \beta_n^{(0)}$ represent the values of the coordinates computed from the elements $\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}$ Suppose the best elements are related to the approximate elements by

$$\alpha_i = \alpha_i^{(0)} + \Delta\alpha_i, \quad (i=1, \ldots, 6)$$

Then

$$(72) \quad \begin{cases} \lambda_i^{(0)} = f_i\,(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}), \\ \beta_i^{(0)} = g_i\,(\alpha_1^{(0)}, \ldots, \alpha_6^{(0)}), \\ \lambda_i = f_i\,(\alpha_1^{(0)} + \Delta\alpha_1, \ldots, \alpha_6^{(0)} + \Delta\alpha_6), \\ \beta_i = g_i\,(\alpha_1^{(0)} + \Delta\alpha_1, \ldots, \alpha_6^{(0)} + \Delta\alpha_6), \quad (i=1, \ldots, n), \end{cases}$$

where the subscripts on f and g indicate that the computations are to be made for the time t_i

The expansions of the last two equations of (72) give

$$(73) \quad \begin{cases} \lambda_i - \lambda_i^{(0)} = \dfrac{\partial f_i}{\partial \alpha_1}\Delta\alpha_1 + \ldots + \dfrac{\partial f_i}{\partial \alpha_6}\Delta\alpha_6 + \text{higher terms}, \\ \beta_i - \beta_i^{(0)} = \dfrac{\partial g_i}{\partial \alpha_1}\Delta\alpha_1 + \ldots + \dfrac{\partial g_i}{\partial \alpha_6}\Delta\alpha_6 + \quad ,, \quad ,, \quad (i=1, \ldots, n) \end{cases}$$

If the partial derivatives were known and the higher terms of these equations were neglected they would become linear in the six unknowns $\alpha_1, \ldots, \alpha_6$, and could be solved by the method of Least Squares in case the number of observations was greater than three The partial derivatives can be computed by direct processes, as the way in which the coördinates depend upon the elements is known, but, as before, it is simpler to calculate them indirectly

Let $\Delta'\alpha_1$ be an arbitrary small variation of α_1, and let the computed values of the coördinates with this element changed be $\lambda_i^{(1)}$ and $\beta_i^{(1)}$ Then

$$(74) \quad \begin{cases} \lambda_i^{(1)} = f_i\,(\alpha_1^{(0)} + \Delta'\alpha_1,\ \alpha_2^{(0)}, \ldots, \alpha_6^{(0)}), \\ \beta_i^{(1)} = g_i\,(\alpha_1^{(0)} + \Delta'\alpha_1,\ \alpha_2^{(0)}, \ldots, \alpha_6^{(0)}) \end{cases}$$

Expanding, neglecting terms of order higher than the first, and solving, it is found that

$$(75) \quad \begin{cases} \dfrac{\partial f_i}{\partial \alpha_1} = \dfrac{\lambda_i^{(1)} - \lambda_i^{(0)}}{\Delta'\alpha_1}, \\ \dfrac{\partial g_i}{\partial \alpha_1} = \dfrac{\beta_i^{(1)} - \beta_i^{(0)}}{\Delta'\alpha_1} \end{cases}$$

Similar equations in $\alpha_2, \alpha_3, \ldots, \alpha_6$ give the values of the other partial derivatives They are to be substituted in (73) and the solution for $\Delta\alpha_1, \ldots, \Delta\alpha_6$ made by the method of Least Squares

239 Comments on the General Problem of Determining Orbits The nature of the assumptions involved in the preceding work, and the difficulty of actually determining rigorously the osculating elements of an orbit, will now be pointed out

In the theories of orbits it is assumed that the position of the observer is known, that is, that the motion of the earth is known This would not be true if the unknown body had a mass large enough to produce sensible perturbations of the motion of the earth The applications of the methods have been to comets and small planets whose masses have been in every instance so small that their disturbing effects have been absolutely inappreciable, but it is conceivable that the solar system may some time encounter a body of great mass

It is also assumed in the theories of orbits that the unknown body is describing sensibly a conic section with respect to the sun, or, in other words, that it is not passing near enough to one of the planets so that its orbit is appreciably perturbed This is an assumption which is by no means certainly true In general, it is very nearly true for the brief interval of time covered by the three observations upon which the preliminary orbit is based, but in the longer intervals used in the differential corrections it is necessary to take into account the perturbations This, of course, complicates the problem greatly, for the perturbations are functions of both the approximate elements and also the small corrections applied to them Hence the small corrections $\Delta\alpha_i$ must be chosen so that the observations shall be best represented when the perturbations, which depend upon the $\Delta\alpha_i$, are taken into account In the case of Uranus before the discovery of Neptune it was found impossible to derive a set of elements which would represent the observations without inadmissible discrepancies, even when the perturbations of all the other known planets were included

If the perturbations were so great, as might be the case if a comet were passing near one of the major planets, that the elements derived from three observations were not even approximately correct, then it would be impossible to apply differential corrections, and the whole process would fail

Instead of employing osculating elements astronomers have found it convenient to use *mean elements*, which are a sort of average of the osculating elements for a long period of time The reason for this is clear For, suppose it is desired to compute the perturbations of a small planet by the methods of Chap IX The length of time during which the results are sensibly correct depends upon the slowness with

which the body deviates from the position given by the osculating elements Suppose the perturbations of the major axis by Jupiter are a maximum at the time the osculating elements are defined This will give a period which will be considerably different from the actual period of the body in any complete revolution, and consequently its actual position will rapidly deviate from that used in estimating the disturbing forces, so that in a short time the method will fail

If the average period for a large number of revolutions were used the deviations would be less extreme, and occurring in one way and then in the other would to a large extent balance each other Similar remarks hold with reference to all of the elements

An important problem is then to determine mean elements If the mean element is the average for any particular time the definition is clear, but the problem at once arises to determine how long a time should be taken If the motion were periodic the answer would be at once that the interval should be one complete period, if the motion is not strictly periodic, the interval might be taken as the least common multiple of the periods of the principal periodic terms. No uniform and consistent method of making this computation has been devised

The general problem of determining an orbit is, for an observer on one of a system of n mutually attracting bodies of unknown masses, to determine their positions, their masses, and their motions If there were no body of relatively great mass, as the sun in the solar system, the problem would be beyond the range of present methods. In any case whether there is a dominant body or not, the complete problem involves all of the difficulties of both perturbations and the determination of the elements of undisturbed orbits, and is immensely more complicated than either of them

XXVIII PROBLEMS

1 Suppose three positions of the body are known as in Art. 236. Show (*a*) that the three positions of the body define the elements of the orbit without using the intervals of time in which the different arcs are described, (*b*) find the explicit formulas for determining the orbit, (*c*) compare their length with the ones developed in this chapter, (*d*) show that the parameter is not well determined, depending upon the ratio of small quantities of the third order.

2 Apply the two methods developed in the text and the one of the last example to an asteroid orbit

3 How many observations are needed to define a circular orbit? Develop a method of finding the elements

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Let the xy-plane be the plane of the ecliptic, ΩP the projection of the orbit upon the celestial sphere, Π the projection of the perihelion point, and P the projection of the position of the planet at the time t In place of π and ϵ, adopt the new elements ω and σ defined by the equations

$$(42)\qquad \begin{cases} \omega = \pi - \Omega, \\ \sigma = -nT \end{cases}$$

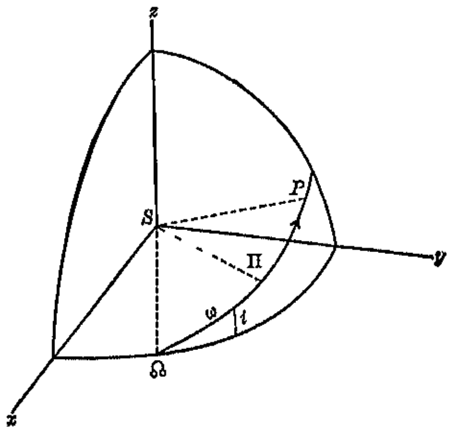

Fig 60.

The following equations are either given in Art. 98, or are obtained from Fig. 60 by the fundamental formulas of Trigonometry:

$$(43)\qquad \begin{cases} n = \dfrac{k\sqrt{S+m_1}}{a^{\frac{3}{2}}}, \\ E - e\sin E = nt + \sigma, \\ r = a(1 - e\cos E), \\ \tan\dfrac{v}{2} = \sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E}{2}, \\ \cos v = \dfrac{\cos E - e}{1 - e\cos E}, \\ \sin v = \dfrac{\sqrt{1-e^2}\sin E}{1 - e\cos E}, \\ x = r\{\cos(v+\omega)\cos\Omega - \sin(v+\omega)\sin\Omega\cos i\}, \\ y = r\{\cos(v+\omega)\sin\Omega + \sin(v+\omega)\cos\Omega\cos i\}, \\ z = r\sin(v+\omega)\sin i \end{cases}$$

From these equations and their derivatives with respect to the time the partial derivatives of the coordinates with respect to the elements can be computed The elements have been chosen in such a manner that they are divided into two groups having distinct properties, Ω, i, and ω define the position of the plane of motion and the orientation of the orbit in the plane, and a, e, and σ define the dimensions and shape of the orbit and the position of the planet in its orbit. Therefore the coordinates in the orbit can be expressed in terms of the elements of the second group alone, and from them, the coordinates in space can be found by means of the first group alone

Take a new system of axes with the origin at the sun, the positive end of the ξ-axis directed to the perihelion point, the η-axis 90° forward in the plane of the orbit, and the ζ-axis perpendicular to the plane of the orbit. Let the direction cosines between the x-axis and the ξ, η, and ζ-axes be α, α', α'', between the y-axis and the ξ, η, and ζ-axes be β, β', β''; and between the z-axis and the ξ, η, and ζ-axes be γ, γ', γ''. Then it follows from Fig 60 that

$$(44)\quad \left\{\begin{aligned} \alpha &= \cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i, \\ \beta &= \cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i, \\ \gamma &= \sin\omega\sin i, \\ \alpha' &= -\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i, \\ \beta' &= -\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i, \\ \gamma' &= \cos\omega\sin i, \\ \alpha'' &= \sin\Omega\sin i, \\ \beta'' &= -\cos\Omega\sin i, \\ \gamma'' &= \cos i \end{aligned}\right.$$

There exist among these nine direction cosines, as can easily be verified, the relations

$$(45)\quad \left\{\begin{array}{lll} \alpha^2 + \beta^2 + \gamma^2 = 1, & & \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0, \\ \alpha'^2 + \beta'^2 + \gamma'^2 = 1, & & \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0, \\ \alpha''^2 + \beta''^2 + \gamma''^2 = 1, & & \alpha''\alpha + \beta''\beta + \gamma''\gamma = 0, \\ \alpha = \beta'\gamma'' - \gamma'\beta'', & \alpha' = \beta''\gamma - \gamma''\beta, & \alpha'' = \beta\gamma' - \gamma\beta', \\ \beta = \gamma'\alpha'' - \alpha'\gamma'', & \beta' = \gamma''\alpha - \alpha''\gamma, & \beta'' = \gamma\alpha' - \alpha\gamma', \\ \gamma = \alpha'\beta'' - \beta'\alpha'', & \gamma' = \alpha''\beta - \beta''\alpha, & \gamma'' = \alpha\beta' - \beta\alpha'. \end{array}\right.$$

It follows from (43) and (44) and the definition of the new system of axes that

$$
(46)\quad \begin{cases}
\xi = r\cos v = a(\cos E - e), \quad \eta = a\sqrt{1-e^2}\sin E, \\[4pt]
\dfrac{\partial E}{\partial t} = \dfrac{n}{1 - e\cos E}, \\[4pt]
\xi' = \dfrac{-\,na\sin E}{1 - e\cos E} = \dfrac{-\,k\sqrt{S+m_1}\sin E}{\sqrt{a}(1 - e\cos E)}, \\[4pt]
\eta' = \dfrac{na\sqrt{1-e^2}\cos E}{1 - e\cos E} = \dfrac{k\sqrt{S+m_1}\sqrt{1-e^2}\cos E}{\sqrt{a}(1 - e\cos E)}, \\[4pt]
x = \alpha\xi + \alpha'\eta, \quad y = \beta\xi + \beta'\eta, \quad z = \gamma\xi + \gamma'\eta, \\
x' = \alpha\xi' + \alpha'\eta', \quad y' = \beta\xi' + \beta'\eta', \quad z' = \gamma\xi' + \gamma'\eta',
\end{cases}
$$

where the accents on x, y, z, ξ, η, and ζ indicate first derivatives with respect to t

The partial derivatives of $\alpha, \cdots, \gamma''$ with respect to the elements may be computed once for all, they are found from (44) to be

$$
(47)\quad \begin{cases}
\dfrac{\partial\alpha}{\partial\omega} = \alpha', & \dfrac{\partial\alpha'}{\partial\omega} = -\,\alpha, & \dfrac{\partial\alpha''}{\partial\omega} = 0, \\[6pt]
\dfrac{\partial\beta}{\partial\omega} = \beta', & \dfrac{\partial\beta'}{\partial\omega} = -\,\beta, & \dfrac{\partial\beta''}{\partial\omega} = 0, \\[6pt]
\dfrac{\partial\gamma}{\partial\omega} = \gamma', & \dfrac{\partial\gamma'}{\partial\omega} = -\,\gamma, & \dfrac{\partial\gamma''}{\partial\omega} = 0,
\end{cases}
$$

$$
(48)\quad \begin{cases}
\dfrac{\partial\alpha}{\partial\Omega} = -\,\beta, & \dfrac{\partial\alpha'}{\partial\Omega} = -\,\beta', & \dfrac{\partial\alpha''}{\partial\Omega} = -\,\beta'', \\[6pt]
\dfrac{\partial\beta}{\partial\Omega} = \alpha, & \dfrac{\partial\beta'}{\partial\Omega} = \alpha', & \dfrac{\partial\beta''}{\partial\Omega} = \alpha'', \\[6pt]
\dfrac{\partial\gamma}{\partial\Omega} = 0, & \dfrac{\partial\gamma'}{\partial\Omega} = 0, & \dfrac{\partial\gamma''}{\partial\Omega} = 0,
\end{cases}
$$

$$
(49)\quad \begin{cases}
\dfrac{\partial\alpha}{\partial i} = \alpha''\sin\omega, & \dfrac{\partial\alpha'}{\partial i} = \alpha''\cos\omega, & \dfrac{\partial\alpha''}{\partial i} = +\sin\Omega\cos i, \\[6pt]
\dfrac{\partial\beta}{\partial i} = \beta''\sin\omega, & \dfrac{\partial\beta'}{\partial i} = \beta''\cos\omega, & \dfrac{\partial\beta''}{\partial i} = -\cos\Omega\cos i, \\[6pt]
\dfrac{\partial\gamma}{\partial i} = \gamma''\sin\omega, & \dfrac{\partial\gamma'}{\partial i} = \gamma''\cos\omega, & \dfrac{\partial\gamma''}{\partial i} = -\sin i.
\end{cases}
$$

There are as many brackets to be computed as there are combinations of the six elements taken two at a time, or $\dfrac{6!}{2!\,4!} = 15$. Three of them involve elements of only the first group, nine, one element of the first group and one of the second, and three, elements of only the second group. Let K and L represent any of the elements of the first group, Ω, i, ω, and P and Q any of the elements of the second group, a, e, σ. Then the Lagrangian brackets to be computed are

$$
(50)\quad \begin{cases}
(a)\quad [K, L] = S\left\{\dfrac{\partial x}{\partial K}\dfrac{\partial x'}{\partial L} - \dfrac{\partial x'}{\partial K}\dfrac{\partial x}{\partial L}\right\}, & \text{(3 equations)},\\[2ex]
(b)\quad [K, P] = S\left\{\dfrac{\partial x}{\partial K}\dfrac{\partial x'}{\partial P} - \dfrac{\partial x'}{\partial K}\dfrac{\partial x}{\partial P}\right\}, & \text{(9 equations)},\\[2ex]
(c)\quad [P, Q] = S\left\{\dfrac{\partial x}{\partial P}\dfrac{\partial x'}{\partial Q} - \dfrac{\partial x'}{\partial P}\dfrac{\partial x}{\partial Q}\right\}, & \text{(3 equations)}.
\end{cases}
$$

It is found from (46) that

$$
(51)\quad \begin{cases}
\dfrac{\partial x}{\partial K} = \xi\dfrac{\partial \alpha}{\partial K} + \eta\dfrac{\partial \alpha'}{\partial K}, & \dfrac{\partial x'}{\partial K} = \xi'\dfrac{\partial \alpha}{\partial K} + \eta'\dfrac{\partial \alpha'}{\partial K},\\[2ex]
\dfrac{\partial x}{\partial P} = \alpha\dfrac{\partial \xi}{\partial P} + \alpha'\dfrac{\partial \eta}{\partial P}, & \dfrac{\partial x'}{\partial P} = \alpha\dfrac{\partial \xi'}{\partial P} + \alpha'\dfrac{\partial \eta'}{\partial P},
\end{cases}
$$

and similar equations in y and z.

216. Computation of $[\omega, \Omega]$, $[\Omega, i]$, $[i, \omega]$. Let S indicate that the sum of the functions, symmetrical in α, β, and γ, is to be taken. Then the first equation of (50) becomes as a consequence of (51)

$$[K, L] = (\xi\eta' - \eta\xi')S\left\{\frac{\partial \alpha}{\partial K}\frac{\partial \alpha'}{\partial L} - \frac{\partial \alpha'}{\partial K}\frac{\partial \alpha}{\partial L}\right\}$$

But the law of areas [Art. 89] gives

$$\xi\eta' - \eta\xi' = \xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} = k\sqrt{(S + m_1)a(1 - e^2)} = na^2\sqrt{1 - e^2}.$$

Therefore

$$(52)\qquad [K, L] = na^2\sqrt{1 - e^2}\,S\left\{\frac{\partial \alpha}{\partial K}\frac{\partial \alpha'}{\partial L} - \frac{\partial \alpha'}{\partial K}\frac{\partial \alpha}{\partial L}\right\}.$$

On computing the right member of this equation by means of (47),

(48), and (49), and reducing by means of (45), the brackets involving elements of only the first group are found to be

$$(53)\quad \begin{cases} [\omega, \Omega] = na^2\sqrt{1-e^2}\,(-\alpha\beta - \alpha'\beta' + \alpha\beta + \alpha'\beta') = 0, \\ [\Omega, i] = na^2\sqrt{1-e^2}\,\{(\alpha\beta'' - \beta\alpha'')\cos\omega \\ \qquad\qquad + (\beta'\alpha'' - \alpha'\beta'')\sin\omega\} \\ \qquad = na^2\sqrt{1-e^2}\,(-\gamma'\cos\omega - \gamma\sin\omega) \\ \qquad = -\,na^2\sqrt{1-e^2}\sin i, \\ [i, \;\omega] = -\,na^2\sqrt{1-e^2}\,\{(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'')\cos\omega \\ \qquad\qquad + (\alpha''\alpha + \beta''\beta + \gamma''\gamma)\sin\omega\} = 0 \end{cases}$$

217. Computation of $[K, P]$. The second equations of (50) become, as a consequence of (51),

$$\begin{aligned} [K, P] = S\Bigg\{&\left[\xi\frac{\partial\alpha}{\partial K} + \eta\frac{\partial\alpha'}{\partial K}\right]\left[\alpha\frac{\partial\xi'}{\partial P} + \alpha'\frac{\partial\eta}{\partial P}\right] \\ &- \left[\xi'\frac{\partial\alpha}{\partial K} + \eta'\frac{\partial\alpha'}{\partial K}\right]\left[\alpha\frac{\partial\xi}{\partial P} + \alpha'\frac{\partial\eta'}{\partial P}\right]\Bigg\} \\ = +&\left[\alpha\frac{\partial\alpha}{\partial K} + \beta\frac{\partial\beta}{\partial K} + \gamma\frac{\partial\gamma}{\partial K}\right]\left[\xi\frac{\partial\xi'}{\partial P} - \xi'\frac{\partial\xi}{\partial P}\right] \\ +&\left[\alpha'\frac{\partial\alpha'}{\partial K} + \beta'\frac{\partial\beta'}{\partial K} + \gamma'\frac{\partial\gamma'}{\partial K}\right]\left[\eta\frac{\partial\eta'}{\partial P} - \eta'\frac{\partial\eta}{\partial P}\right] \\ +&\left[\alpha\frac{\partial\alpha'}{\partial K} + \beta\frac{\partial\beta'}{\partial K} + \gamma\frac{\partial\gamma'}{\partial K}\right]\left[\eta\frac{\partial\xi'}{\partial P} - \eta'\frac{\partial\xi}{\partial P}\right] \\ +&\left[\alpha'\frac{\partial\alpha}{\partial K} + \beta'\frac{\partial\beta}{\partial K} + \gamma'\frac{\partial\gamma}{\partial K}\right]\left[\xi\frac{\partial\eta'}{\partial P} - \xi'\frac{\partial\eta}{\partial P}\right]. \end{aligned}$$

It follows from equations (45), (47), (48), and (49) that

$$\begin{aligned} \alpha\frac{\partial\alpha}{\partial K} + \beta\frac{\partial\beta}{\partial K} + \gamma\frac{\partial\gamma}{\partial K} &= 0, \\ \alpha'\frac{\partial\alpha'}{\partial K} + \beta'\frac{\partial\beta'}{\partial K} + \gamma'\frac{\partial\gamma'}{\partial K} &= 0, \\ \alpha\frac{\partial\alpha'}{\partial K} + \beta\frac{\partial\beta'}{\partial K} + \gamma\frac{\partial\gamma'}{\partial K} &= -\left[\alpha'\frac{\partial\alpha}{\partial K} + \beta'\frac{\partial\beta}{\partial K} + \gamma'\frac{\partial\gamma}{\partial K}\right]. \end{aligned}$$

Therefore

$$
(54)\quad \begin{cases} [K, P] = \left[\alpha' \dfrac{\partial \alpha}{\partial K} + \beta' \dfrac{\partial \beta}{\partial K} + \gamma' \dfrac{\partial \gamma}{\partial K} \right] \\ \qquad\qquad \times \left[\xi \dfrac{\partial \eta'}{\partial P} + \eta' \dfrac{\partial \xi}{\partial P} - \xi' \dfrac{\partial \eta}{\partial P} - \eta \dfrac{\partial \xi'}{\partial P} \right] \\ \qquad = \left[\alpha' \dfrac{\partial \alpha}{\partial K} + \beta' \dfrac{\partial \beta}{\partial K} + \gamma' \dfrac{\partial \gamma}{\partial K} \right] \dfrac{\partial (\xi\eta' - \eta\xi')}{\partial P} \\ \qquad = k\sqrt{S + m_1} \left[\alpha' \dfrac{\partial \alpha}{\partial K} + \beta' \dfrac{\partial \beta}{\partial K} + \gamma' \dfrac{\partial \gamma}{\partial K} \right] \dfrac{\partial \sqrt{p}}{\partial P} \end{cases}
$$

Let $P = a, e, \sigma$ in succession Then it is found that

$$
(55)\quad \begin{cases} k\sqrt{S + m_1} \dfrac{\partial \sqrt{a(1 - e^2)}}{\partial a} = \dfrac{na}{2}\sqrt{1 - e^2}, \\ k\sqrt{S + m_1} \dfrac{\partial \sqrt{a(1 - e^2)}}{\partial e} = -\dfrac{na^2 e}{\sqrt{1 - e^2}}, \\ k\sqrt{S + m_1} \dfrac{\partial \sqrt{a(1 - e^2)}}{\partial \sigma} = 0 \end{cases}
$$

Let $K = \omega, \Omega, i$ in turn in (54), and make use of (55), then it is found that

$$
(56)\quad \begin{cases} [\omega, a] = \dfrac{na}{2}\sqrt{1 - e^2}, & [\omega, e] = \dfrac{-na^2 e}{\sqrt{1 - e^2}}, & [\omega, \sigma] = 0, \\ [\Omega, a] = \dfrac{na}{2}\sqrt{1 - e^2}\cos i, & [i, a] = 0, & [i, e] = 0, \\ [\Omega, e] = \dfrac{-na^2 e}{\sqrt{1 - e^2}}\cos i, & [\Omega, \sigma] = 0, & [i, \sigma] = 0 \end{cases}
$$

218. Computation of $[a, e]$, $[e, \sigma]$, $[\sigma, a]$. The third equation of (50) becomes, as a consequence of (51),

$$
\begin{aligned} [P, Q] = S \Big\{ & \left[\alpha \frac{\partial \xi}{\partial P} + \alpha' \frac{\partial \eta}{\partial P} \right] \left[\alpha \frac{\partial \xi'}{\partial Q} + \alpha' \frac{\partial \eta'}{\partial Q} \right] \\ & - \left[\alpha \frac{\partial \xi'}{\partial P} + \alpha' \frac{\partial \eta'}{\partial P} \right] \left[\alpha \frac{\partial \xi}{\partial Q} + \alpha' \frac{\partial \eta}{\partial Q} \right] \Big\} \\ = & + (\alpha^2 + \beta^2 + \gamma^2) \left[\frac{\partial \xi}{\partial P}\frac{\partial \xi'}{\partial Q} - \frac{\partial \xi}{\partial Q}\frac{\partial \xi'}{\partial P} \right] \\ & + (\alpha'^2 + \beta'^2 + \gamma'^2) \left[\frac{\partial \eta}{\partial P}\frac{\partial \eta'}{\partial Q} - \frac{\partial \eta}{\partial Q}\frac{\partial \eta'}{\partial P} \right] \\ & + (\alpha\alpha' + \beta\beta' + \gamma\gamma') \left[\frac{\partial \xi}{\partial P}\frac{\partial \eta'}{\partial Q} - \frac{\partial \xi}{\partial Q}\frac{\partial \eta'}{\partial P} + \frac{\partial \xi'}{\partial Q}\frac{\partial \eta}{\partial P} - \frac{\partial \xi'}{\partial P}\frac{\partial \eta}{\partial Q} \right]. \end{aligned}
$$

As a consequence of equations (45), the right member of this equation reduces to

$$(57)\qquad [P, Q] = \frac{\partial \xi}{\partial P}\frac{\partial \xi'}{\partial Q} - \frac{\partial \xi}{\partial Q}\frac{\partial \xi'}{\partial P} + \frac{\partial \eta}{\partial P}\frac{\partial \eta'}{\partial Q} - \frac{\partial \eta}{\partial Q}\frac{\partial \eta'}{\partial P}$$

Since the brackets do not contain the time explicitly t may be given any value after the partial derivatives have been formed The partial derivatives become the simplest when $t = T$, the time of perihelion passage. For this value of t, $E = 0$, $r = a(1 - e)$, and it is found from equations (46) that*

$$(58)\quad \begin{cases} \dfrac{\partial \xi}{\partial a} = 1 - e, & \dfrac{\partial \eta}{\partial a} = 0, \quad \dfrac{\partial \xi'}{\partial a} = 0, \quad \dfrac{\partial \eta'}{\partial a} = -\dfrac{n}{2}\sqrt{\dfrac{1+e}{1-e}}, \\[2ex] \dfrac{\partial \xi}{\partial e} = -a, & \dfrac{\partial \eta}{\partial e} = 0, \quad \dfrac{\partial \xi'}{\partial e} = 0, \quad \dfrac{\partial \eta'}{\partial e} = \dfrac{1}{1-e}\cdot\dfrac{na}{\sqrt{1-e^2}}, \\[2ex] \dfrac{\partial \xi}{\partial \sigma} = 0, & \dfrac{\partial \eta}{\partial \sigma} = a\sqrt{\dfrac{1+e}{1-e}}, \quad \dfrac{\partial \xi'}{\partial \sigma} = \dfrac{-na}{(1-e)^2}, \quad \dfrac{\partial \eta'}{\partial \sigma} = 0. \end{cases}$$

Then equation (57) gives

$$(59)\qquad [a, e] = 0, \qquad [e, \sigma] = 0, \qquad [\sigma, a] = \frac{na}{2}$$

On making use of the fact that $[\alpha_i, \alpha_j] = -[\alpha_j, \alpha_i]$ and equations (53), (56), and (59), equations (33) become

$$(60)\quad \begin{cases} \dfrac{na}{2}\sqrt{1-e^2}\dfrac{da}{dt} - \dfrac{na^2 e}{\sqrt{1-e^2}}\dfrac{de}{dt} = m_2\dfrac{\partial R_{1,2}}{\partial \omega}, \\[2ex] -na^2\sqrt{1-e^2}\sin i\dfrac{di}{dt} + \dfrac{na}{2}\sqrt{1-e^2}\cos i\dfrac{da}{dt} \\[1ex] \qquad\qquad - \dfrac{na^2 e}{\sqrt{1-e^2}}\cos i\dfrac{de}{dt} = m_2\dfrac{\partial R_{1,}}{\partial \Omega}, \\[2ex] na^2\sqrt{1-e^2}\sin i\dfrac{d\Omega}{dt} = m_2\dfrac{\partial R_{1,2}}{\partial i}, \end{cases}$$

* It should be remembered that a and e enter explicitly and also implicitly through E and n, for E is defined by the equation

$$E - e\sin E = n(t - T) = \frac{k\sqrt{S + m_1}}{a^{\frac{3}{2}}}(t - T).$$

Then, e g, $\dfrac{\partial \xi}{\partial a} = \cos E - e - a\sin E\dfrac{\partial E}{\partial a} = 1 - e$ when $t = T$, etc

$$
(60)\quad \begin{cases}
-\dfrac{na}{2}\sqrt{1-e^2}\dfrac{d\omega}{dt}-\dfrac{na}{2}\sqrt{1-e^2}\cos\imath\dfrac{d\Omega}{\partial t}-\dfrac{na}{2}\dfrac{d\sigma}{dt}=m_2\dfrac{\partial R_{1,2}}{\partial a},\\[2ex]
\dfrac{na^2e}{\sqrt{1-e^2}}\dfrac{d\omega}{dt}+\dfrac{na^2e\cos\imath}{\sqrt{1-e^2}}\dfrac{d\Omega}{dt}=m_2\dfrac{\partial R_{1,2}}{\partial e},\\[2ex]
\dfrac{na}{2}\dfrac{da}{dt}=m_2\dfrac{\partial R_{1,2}}{\partial\sigma}
\end{cases}
$$

These equations are easily solved for the derivatives, and give

$$
(61)\quad \begin{cases}
\dfrac{d\Omega}{dt}=\dfrac{m_2}{na^2\sqrt{1-e^2}\sin\imath}\dfrac{\partial R_{1,2}}{\partial\imath},\\[2ex]
\dfrac{d\imath}{dt}=\dfrac{m_2\cos\imath}{na^2\sqrt{1-e^2}\sin\imath}\dfrac{\partial R_{1,2}}{\partial\omega}-\dfrac{m_2}{na^2\sqrt{1-e^2}\sin\imath}\dfrac{\partial R_{1,2}}{\partial\Omega},\\[2ex]
\dfrac{d\omega}{dt}=\dfrac{-m_2\cos\imath}{na^2\sqrt{1-e^2}\sin\imath}\dfrac{\partial R_{1,2}}{\partial\imath}+\dfrac{m_2\sqrt{1-e^2}}{na^2e}\dfrac{\partial R_{1,2}}{\partial e},\\[2ex]
\dfrac{da}{dt}=\dfrac{2m_2}{na}\dfrac{\partial R_{1,2}}{\partial\sigma},\\[2ex]
\dfrac{de}{dt}=\dfrac{m_2(1-e^2)}{na^2e}\dfrac{\partial R_{1,2}}{\partial\sigma}-\dfrac{m_2\sqrt{1-e^2}}{na^2e}\dfrac{\partial R_{1,2}}{\partial\omega},\\[2ex]
\dfrac{\partial\sigma}{dt}=-\dfrac{m_2(1-e^2)}{na^2e}\dfrac{\partial R_{1,2}}{\partial e}-\dfrac{2m_2}{na}\dfrac{\partial R_{1,2}}{\partial a}.
\end{cases}
$$

The perturbative function $R_{1,2}$ involves the element a explicitly, and also implicitly through n which enters only in the combination $nt+\sigma$. Consequently the last equation of (61) becomes

$$
(62)\quad \frac{d\sigma}{dt}=-\frac{m_2(1-e^2)}{na^2e}\frac{\partial R_{1,2}}{\partial e}-\frac{2m_2}{na}\left(\frac{\partial R_{1,2}}{\partial a}\right)-\frac{2m_2}{na}\frac{\partial R_{1,2}}{\partial n}\frac{\partial n}{\partial a},
$$

where the partial derivative in parenthesis indicates the derivative is taken only so far as the parameter appears explicitly.

It follows from the combination $nt+\sigma$ that

$$
(63)\qquad \frac{2m_2}{na}\frac{\partial R_{1,2}}{\partial n}=\frac{2m_2t}{na}\frac{\partial R_{1,2}}{\partial\sigma}=t\frac{da}{dt}
$$

It will be shown [Arts. 225–227] that $\dfrac{\partial R_{1,2}}{\partial\sigma}$ is a sum of periodic terms; therefore σ, as defined by (62), contains terms which are the products of t and trigonometric terms It is obvious that such an element is inconvenient when large values of t are to be used.

In order to avoid this difficulty Leverrier used* in place of σ the mean longitude from the perihelion as an element. It is defined by

$$l = \int n dt + \sigma, \tag{64}$$

whence

$$\frac{dl}{dt} = n + t\frac{dn}{dt} + \frac{d\sigma}{dt}. \tag{65}$$

Since $n = \dfrac{k\sqrt{S+m_1}}{a^{\frac{3}{2}}}$, it follows that

$$\frac{\partial n}{\partial a} = -\frac{3}{2}\frac{n}{a}, \qquad \frac{dn}{dt} = -\frac{3n}{2a}\frac{da}{dt}. \tag{66}$$

Therefore equation (65) becomes, on making use of (62),

$$\frac{dl}{dt} = n - \frac{m_2(1-e^2)}{na^2e}\frac{\partial R_{1,2}}{\partial e} - \frac{2m_2}{na}\left(\frac{\partial R_{1,2}}{\partial a}\right). \tag{67}$$

Since $\dfrac{\partial R_{1,2}}{\partial \sigma} = \dfrac{\partial R_{1,2}}{\partial l}$, the fourth and fifth equations, where alone the partial derivative of $R_{1,2}$ with respect to σ occurs, will not be changed in form. Hence, if l is used in place of σ throughout (61), the equations will be unchanged in form, and the partial derivative of $R_{1,2}$ with respect to a is to be taken only so far as a occurs explicitly.

219. Change from Ω, ω, and σ to Ω, π, and ϵ. The transformation from the elements Ω, ω, and σ to Ω, π, and ϵ is readily made because the relations between the ω and σ and the π and ϵ are very simple. It follows from the definitions of Arts. 214 and 215 that

$$\begin{cases} \Omega = \Omega, \\ \omega = \pi - \Omega, \\ \sigma = \epsilon - \pi; \end{cases} \tag{68}$$

whence

$$\begin{cases} \dfrac{d\Omega}{dt} = \dfrac{d\Omega}{dt}, \\[2ex] \dfrac{d\omega}{dt} = \dfrac{d\pi}{dt} - \dfrac{d\Omega}{dt}, \\[2ex] \dfrac{d\sigma}{dt} = \dfrac{d\epsilon}{dt} - \dfrac{d\pi}{dt}. \end{cases} \tag{69}$$

On solving (68) for Ω, π, and ϵ in terms of Ω, ω, and σ, it is found that

* *Annales de l'Observatoire de Paris*, vol. I, p. 255.

$$(70)\qquad \begin{cases} \Omega = \Omega, \\ \pi = \omega + \Omega, \\ \epsilon = \sigma + \pi = \sigma + \omega + \Omega. \end{cases}$$

Hence the transformations in the partial derivatives are given by the equations

$$(71)\qquad \begin{cases} \dfrac{\partial R_{1,2}}{\partial \Omega} = \left(\dfrac{\partial R_{1,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \Omega} + \left(\dfrac{\partial R_{1,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \Omega} + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \Omega} \\ \qquad\quad = \left(\dfrac{\partial R_{1,2}}{\partial \Omega}\right) + \left(\dfrac{\partial R_{1,2}}{\partial \pi}\right) + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right), \\ \dfrac{\partial R_{1,2}}{\partial \omega} = \left(\dfrac{\partial R_{1,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \omega} + \left(\dfrac{\partial R_{1,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \omega} + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \omega} \\ \qquad\quad = \left(\dfrac{\partial R_{1,2}}{\partial \pi}\right) + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right), \\ \dfrac{\partial R_{1,2}}{\partial \sigma} = \left(\dfrac{\partial R_{1,2}}{\partial \Omega}\right)\dfrac{\partial \Omega}{\partial \sigma} + \left(\dfrac{\partial R_{1,2}}{\partial \pi}\right)\dfrac{\partial \pi}{\partial \sigma} + \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right)\dfrac{\partial \epsilon}{\partial \sigma} \\ \qquad\quad = \left(\dfrac{\partial R_{1,2}}{\partial \epsilon}\right) \end{cases}$$

On substituting (69) and (71) in (61) and omitting the parentheses around the partial derivatives, and on solving for the derivatives of the elements with respect to t, it is found that

$$(72)\qquad \begin{cases} \dfrac{d\Omega}{dt} = \dfrac{m_2}{na^2\sqrt{1-e^2}\sin i}\dfrac{\partial R_{1,2}}{\partial i}, \\[2ex] \dfrac{di}{dt} = \dfrac{-m_2}{na^2\sqrt{1-e^2}\sin i}\dfrac{\partial R_{1,2}}{\partial \Omega} - \dfrac{m_2 \tan\dfrac{i}{2}}{na^2\sqrt{1-e^2}}\left[\dfrac{\partial R_{1,2}}{\partial \pi} + \dfrac{\partial R_{1,2}}{\partial \epsilon}\right], \\[2ex] \dfrac{d\pi}{dt} = \dfrac{m_2 \tan\dfrac{i}{2}}{na^2\sqrt{1-e^2}}\dfrac{\partial R_{1,2}}{\partial i} + \dfrac{m_2\sqrt{1-e^2}}{na^2 e}\dfrac{\partial R_{1,2}}{\partial e}, \\[2ex] \dfrac{da}{dt} = \dfrac{2m_2}{na}\dfrac{\partial R_{1,2}}{\partial \epsilon}, \\[2ex] \dfrac{de}{dt} = -m_2\sqrt{1-e^2}\,\dfrac{1-\sqrt{1-e^2}}{na^2 e}\dfrac{\partial R_{1,2}}{\partial \epsilon} - \dfrac{m_2\sqrt{1-e^2}}{na^2 e}\dfrac{\partial R_{1,2}}{\partial \pi}, \\[2ex] \dfrac{d\epsilon}{dt} = \dfrac{m_2 \tan\dfrac{i}{2}}{na^2\sqrt{1-e^2}}\dfrac{\partial R_{1,2}}{\partial i} + m_2\sqrt{1-e^2}\,\dfrac{1-\sqrt{1-e^2}}{na^2 e}\dfrac{\partial R_{1,2}}{\partial e} \\ \qquad\qquad - \dfrac{2m_2}{na}\dfrac{\partial R_{1,2}}{\partial a} \end{cases}$$

These equations,* together with the corresponding ones for the elements of the planet m_2, constitute a rigorous system of differential equations for the determination of the motion of the planets m_1 and m_2 with respect to the sun when there are no other forces than the mutual attractions of the three bodies

If $R_{1,2}$ is expressed in terms of the time and the osculating elements at the epoch t_0, equations (72) become the explicit expressions for the first half of the system (27), and define the perturbations of the elements which are of the first order with respect to the masses

220. Introduction of Rectangular Components of the Disturbing Acceleration. Equations (72) require for their application that $R_{1,2}$ shall be expressed first in terms of the elements, after which the partial derivatives must be formed In some cases, especially in the orbits of comets, it is advantageous to have the rates of variation of the elements expressed in terms of three rectangular components of the disturbing acceleration

The disturbing acceleration will be resolved into three rectangular components W, S, R, where W is the component of acceleration perpendicular to the plane of the orbit with the positive direction toward the north pole, S is the component in the plane of the orbit which acts at right angles to the radius vector with the positive direction making an angle less than 90° with the direction of motion, R is the component acting along the radius vector with the positive direction away from the sun. The components used in the preceding chapter evidently might be employed here instead of these, but the resulting equations would be less simple

In order to obtain the desired equations it is only necessary to express the partial derivatives of $R_{1,2}$ with respect to the elements in terms of W, S, and R, and to substitute them in (61) or (72), depending upon the set of elements used The transformation will be made for the elements used in equations (61)

The quantities $m_2\dfrac{\partial R_{1,2}}{\partial x}$, $m_2\dfrac{\partial R_{1,2}}{\partial y}$, $m_2\dfrac{\partial R_{1,2}}{\partial z}$ are the components of the disturbing acceleration parallel to the fixed axes of reference. It follows from the elementary properties of the

* The subscript 1, which was omitted from the coordinates and elements in Art 213, should be replaced when the equations for more than one planet are written

resolution and composition of accelerations that $m_2 \dfrac{\partial R_{1,2}}{\partial x}$ is equal to the sum of the projections of W, S, and R upon the x-axis, and similarly for the others

Let u represent the argument of the latitude, or the distance from the ascending node to the planet P, Fig 61 Then it follows

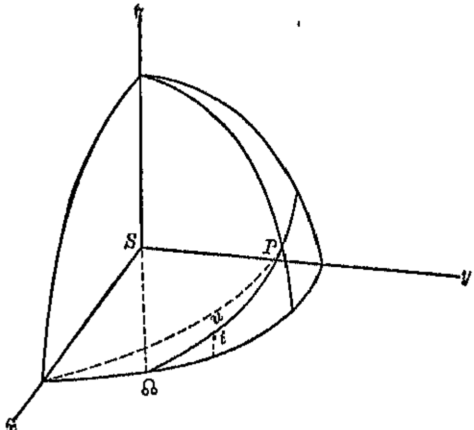

Fig 61

from the fundamental formulas of Trigonometry that

$$(73)\quad \begin{cases} m_2 \dfrac{\partial R_{1,2}}{\partial x} = + R(\cos u \cos \Omega - \sin u \sin \Omega \cos i) \\ \qquad\qquad - S(\sin u \cos \Omega + \cos u \sin \Omega \cos i) \\ \qquad\qquad + W \sin \Omega \sin i, \\ m_2 \dfrac{\partial R_{1,2}}{\partial y} = + R(\cos u \sin \Omega + \sin u \cos \Omega \cos i) \\ \qquad\qquad - S(\sin u \sin \Omega - \cos u \cos \Omega \cos i) \\ \qquad\qquad - W \cos \Omega \sin i, \\ m_2 \dfrac{\partial R_{1,2}}{\partial z} = + R \sin u \sin i + S \cos u \sin i + W \cos i \end{cases}$$

Let s represent any of the elements $\Omega, \cdots, \sigma$, then

$$(74)\qquad \frac{\partial R_{1,2}}{\partial s} = \frac{\partial R_{1,2}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial R_{1,2}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial R_{1,2}}{\partial z}\frac{\partial z}{\partial s}.$$

The derivatives $\dfrac{\partial R_{1,2}}{\partial x}, \dfrac{\partial R_{1,2}}{\partial y}, \dfrac{\partial R_{1,2}}{\partial z}$ are given in (73) and when $\dfrac{\partial x}{\partial s}, \dfrac{\partial y}{\partial s}$, and $\dfrac{\partial z}{\partial s}$ have been found, the transformation can be completed at once.

It follows from equations (51) that

$$
(75)\quad \begin{cases}
\dfrac{\partial x}{\partial K} = \xi \dfrac{\partial \alpha}{\partial K} + \eta \dfrac{\partial \alpha'}{\partial K}, & \dfrac{\partial x}{\partial P} = \alpha \dfrac{\partial \xi}{\partial P} + \alpha' \dfrac{\partial \eta}{\partial P}, \\[2ex]
\dfrac{\partial y}{\partial K} = \xi \dfrac{\partial \beta}{\partial K} + \eta \dfrac{\partial \beta'}{\partial K}, & \dfrac{\partial y}{\partial P} = \beta \dfrac{\partial \xi}{\partial P} + \beta' \dfrac{\partial \eta}{\partial P}, \\[2ex]
\dfrac{\partial z}{\partial K} = \xi \dfrac{\partial \gamma}{\partial K} + \eta \dfrac{\partial \gamma'}{\partial K}, & \dfrac{\partial z}{\partial P} = \gamma \dfrac{\partial \xi}{\partial P} + \gamma' \dfrac{\partial \eta}{\partial P},
\end{cases}
$$

where K is any of the elements Ω, i, ω, and P any of the elements a, e, σ The quantities α, , γ' are defined in (44), and their derivatives are given in (47), (48), and (49), the derivatives $\dfrac{\partial \xi}{\partial P}$ and $\dfrac{\partial \eta}{\partial P}$ are to be computed from (46).

It is found after some rather long but simple reductions that

$$
(76)\quad \begin{cases}
m_2 \dfrac{\partial R_{1,2}}{\partial \Omega} = Sr \cos i - Wr \cos u \sin i, \\[2ex]
m_2 \dfrac{\partial R_{1,2}}{\partial i} = Wr \sin u, \\[2ex]
m_2 \dfrac{\partial R_{1,2}}{\partial \omega} = Sr, \\[2ex]
m_2 \dfrac{\partial R_{1,2}}{\partial a} = R \dfrac{r}{a}, \\[2ex]
m_2 \dfrac{\partial R_{1,2}}{\partial e} = -Ra \cos v + S \left[1 + \dfrac{r}{p} \right] a \sin v, \\[2ex]
m_2 \dfrac{\partial R_{1,2}}{\partial \sigma} = \dfrac{Rae}{\sqrt{1-e^2}} \sin v + S \dfrac{a^2}{r} \sqrt{1-e^2}
\end{cases}
$$

Therefore equations (61) become

$$
(77)\quad \begin{cases}
\dfrac{d\Omega}{dt} = \dfrac{r \sin u}{na^2 \sqrt{1-e^2} \sin i} W, \\[2ex]
\dfrac{di}{dt} = \dfrac{r \cos u}{na^2 \sqrt{1-e^2}} W, \\[2ex]
\dfrac{d\omega}{dt} = \dfrac{-\sqrt{1-e^2} \cos v}{nae} R + \dfrac{\sqrt{1-e^2}}{nae} \left[1 + \dfrac{r}{p} \right] \sin v S \\[2ex]
\qquad\qquad - \dfrac{r \sin u \cot i}{na^2 \sqrt{1-e^2}} W,
\end{cases}
$$

$$
(77)\quad \begin{cases}
\dfrac{da}{dt} = \dfrac{2e \sin v}{n\sqrt{1-e^2}} R + \dfrac{2a\sqrt{1-e^2}}{nr} S, \\[2ex]
\dfrac{de}{dt} = \dfrac{\sqrt{1-e^2}\sin v}{na} R + \dfrac{\sqrt{1-e^2}}{na^2 e}\left[\dfrac{a^2(1-e^2)}{r} - r\right] S, \\[2ex]
\dfrac{d\sigma}{dt} = -\dfrac{1}{na}\left[\dfrac{2r}{a} - \dfrac{1-e^2}{e}\cos v\right] R \\[2ex]
\qquad\qquad - \dfrac{(1-e^2)}{nae}\left[1 + \dfrac{r}{p}\right]\sin v\, S
\end{cases}
$$

XXVI PROBLEMS

1 Find the components S and R of this chapter in terms of T and N, which were used in chapter IX, Art 174

$$
Ans\quad \begin{cases}
S = \dfrac{(1 + e\cos v)}{\sqrt{1+e^2+2e\cos v}} T + \dfrac{e \sin v}{\sqrt{1+e^2+2e\cos v}} N, \\[2ex]
R = \dfrac{e \sin v}{\sqrt{1+e^2+2e\cos v}} T - \dfrac{1 + e\cos v}{\sqrt{1+e^2+2e\cos v}} N
\end{cases}
$$

2 By means of the equations of problem 1 express the variations of the elements Ω, , σ in terms of T and N, and verify all the results contained in the Table of Art 182

3 Explain why $\dfrac{d\omega}{dt}$ contains a term depending upon W

4 Suppose the disturbed body moves in a resisting medium, find the equations for the variations of the elements

$$
Ans\quad \begin{cases}
\dfrac{d\Omega}{dt} = 0, \\[2ex]
\dfrac{di}{dt} = 0, \\[2ex]
\dfrac{d\omega}{dt} = \dfrac{2\sqrt{1-e^2}}{nae}\,\dfrac{\sin v}{\sqrt{1+e^2+2e\cos v}} T, \\[2ex]
\dfrac{da}{dt} = \dfrac{2\sqrt{1+e^2+2e\cos v}}{n\sqrt{1-e^2}} T, \\[2ex]
\dfrac{de}{dt} = \dfrac{2\sqrt{1-e^2}\,(\cos v + e)}{na\sqrt{1+e^2+2e\cos v}} T, \\[2ex]
\dfrac{d\sigma}{dt} = -\dfrac{2(1-e^2)(1+e^2+e\cos v)\sin v}{nae(1+e\cos v)\sqrt{1+e^2+2e\cos v}} T
\end{cases}
$$

5 Discuss the way in which the elements vary in the last problem, including the values of v for which the maxima and minima in their rates of change occur, when T is a constant, and when it varies as the square of the velocity.

6. Derive the equations corresponding to (77) for the elements Ω, i, π, a, e, and ϵ

$$
Ans \quad \left\{
\begin{aligned}
\frac{d\Omega}{dt} &= \frac{r \sin u}{na\sqrt{1-e^2}\sin i} W, \\
\frac{di}{dt} &= \frac{r \cos u}{na^2\sqrt{1-e^2}} W, \\
\frac{d\pi}{dt} &= 2\sin^2\frac{i}{2}\frac{d\Omega}{dt} + \frac{\sqrt{1-e^2}}{nae}\left\{-R\cos v + S\left(1+\frac{r}{p}\right)\sin v\right\}, \\
\frac{da}{dt} &= \frac{2}{n\sqrt{1-e^2}}\left(Re\sin v + S\frac{p}{r}\right), \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{na}\left\{R\sin v + S\left(\frac{e+\cos v}{1+e\cos v} + \cos v\right)\right\}, \\
\frac{d\epsilon}{dt} &= -\frac{2rR}{na^2} + \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\pi}{dt} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt}.
\end{aligned}
\right.
$$

221 Development of the Perturbative Function. In order to apply equations (72) the perturbative function $R_{1,2}$ must be developed explicitly in terms of the elements and the time From this point on only perturbations of the first order will be considered, therefore, in accordance with the results of Art. 208, the elements which appear in $R_{1,2}$ are the osculating elements at the time t_0

In the notation of Art 205 the perturbative function is

$$
(78) \qquad \left\{
\begin{aligned}
R_{1,2} &= k^2\left[\frac{1}{r_{1,2}} - \frac{x_1x_2 + y_1y_2 + z_1z_2}{r_2^3}\right], \\
r_{1,2} &= \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}, \\
r_2 &= \sqrt{x_2^2 + y_2^2 + z_2^2}.
\end{aligned}
\right.
$$

The perturbing forces evidently depend upon the mutual inclinations of the orbits, rather than upon their inclinations independently to the fixed plane of reference. It will be convenient, therefore, to develop $R_{1,2}$ in terms of the mutual inclination Since this angle is expressible in terms of i_1, i_2, Ω_1, and Ω_2, the partial derivatives of $R_{1,2}$ with respect to these elements will depend in part on their occurring implicitly in this angle.

The development of the perturbative function consists of three steps:*

* There are many more or less important variations of the method outlined here, which is based on the work of Leverrier in the *Annales de l'Observatoire de Paris*, vol I

(*a*) Development of $R_{1,2}$ as a power series in the square of the sine of half the mutual inclination of the orbits.

(*b*) Development of the coefficients of the series obtained in (*a*) into power series in e_1 and e_2.

(*c*) Development of the coefficients of the preceding series into Fourier series in the mean longitudes of the two planets and the angular variables π_1, π_2, Ω_1, and Ω_2.

In the little space available here it will not be possible to give more than a general outline of the operations which are necessary to effect the complete development. A detailed discussion is given in Tisserand's *Mécanique Céleste*, vol. I., chapters XII. to XVIII. inclusive.

222. (*a*) Development of $R_{1,2}$ in the Mutual Inclination. Let S represent the angle between the radii r_1 and r_2; then

$$\frac{1}{r_{1,2}} = (r_1^2 + r_2^2 - 2r_1r_2 \cos S)^{-\frac{1}{2}}. \tag{79}$$

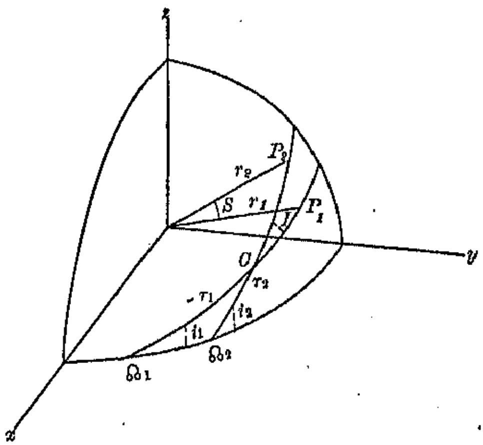

Fig. 62.

Let the angles between r_1 and the x, y, and z-axes be α_1, β_1, γ_1 respectively, and in the case of r_2, α_2, β_2, and γ_2. Then it follows that

$$x_1 = r_1 \cos \alpha_1, \qquad y_1 = r_1 \cos \beta_1, \qquad z_1 = r_1 \cos \gamma_1, \quad \text{etc.,} \tag{80}$$

and

$$\begin{aligned} x_1x_2 + y_1y_2 + z_1z_2 = r_1r_2(\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 \\ + \cos \gamma_1 \cos \gamma_2) = r_1r_2 \cos S. \end{aligned} \tag{81}$$

Let I represent the angle between the two orbits, and τ_1 and τ_2

the distances from their ascending nodes to their point of intersection. From the spherical triangle P_1P_2C the value of $\cos S$ is found to be

$$
(82)\quad \begin{cases}
\cos S = \cos(u_1 - \tau_1)\cos(u_2 - \tau_2) \\
\qquad\qquad + \sin(u_1 - \tau_1)\sin(u_2 - \tau_2)\cos I, \quad \text{or} \\
\cos S = \cos(u_1 - u_2 + \tau_2 - \tau_1) \\
\qquad\qquad - 2\sin(u_1 - \tau_1)\sin(u_2 - \tau_2)\sin^2\dfrac{I}{2}, \\
u_1 - \tau_1 = v_1 + \pi_1 - \Omega_1 - \tau_1, \\
u_2 - \tau_2 = v_2 + \pi_2 - \Omega_2 - \tau_2.
\end{cases}
$$

The quantities I, τ_1, and τ_2 are determined by the formulas of Gauss applied to the triangle $\Omega_1\Omega_2C$

$$
(83)\quad \begin{cases}
\sin I \sin\tau_1 = \sin i_2 \sin(\Omega_1 - \Omega_2), \\
\sin I \sin\tau_2 = \sin i_1 \sin(\Omega_1 - \Omega_2), \\
\sin I \cos\tau_1 = \sin i_1 \cos i_2 - \cos i_1 \sin i_2 \cos(\Omega_1 - \Omega_2), \\
\sin I \cos\tau_2 = -\cos i_1 \sin i_2 + \sin i_1 \cos i_2 \cos(\Omega_1 - \Omega_2), \\
\qquad \cos I = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_1 - \Omega_2).
\end{cases}
$$

For simplicity I, τ_1, and τ_2 will be retained, but it must be remembered when the partial derivatives of $R_{1,2}$ are taken that they are functions of i_1, i_2, Ω_1, and Ω_2.

As a consequence of (79), (81), and (82), the perturbative function can be written in the form

$$
(84)\quad \begin{cases}
R_{1,2} = [r_1^2 + r_2^2 - 2r_1r_2\cos(u_1 - u_2 + \tau_2 - \tau_1)]^{-\frac{1}{2}} \\
\qquad \times \left[1 + \dfrac{4r_1r_2\sin(u_1 - \tau_1)\sin(u_2 - \tau_2)\sin^2\dfrac{I}{2}}{r_1^2 + r_2^2 - 2r_1r_2\cos(u_1 - u_2 + \tau_2 - \tau_1)}\right]^{-\frac{1}{2}} \\
\qquad - \dfrac{r_1}{r_2^2}\Big[\cos(u_1 - u_2 + \tau_2 - \tau_1) \\
\qquad\qquad - 2\sin(u_1 - \tau_1)\sin(u_2 - \tau_2)\sin^2\dfrac{I}{2}\Big].
\end{cases}
$$

The radii r_1 and r_2 are independent of I. The second factor of the first term of the right member of this equation can be expanded by the binomial theorem into an absolutely converging power series in $\sin^2\dfrac{I}{2}$ so long as the numerical value of

$$(85)\qquad \frac{4r_1r_2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\dfrac{I}{2}}{r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)}$$

is less than unity. This fraction is less than, or at most equal to,

$$(86)\qquad \frac{4r_1r_2\sin^2\dfrac{I}{2}}{(r_1-r_1)^2}.$$

If this expression is less than unity for all the values which r_1 and r_2 can take in the given ellipses the expansion of (84) is valid for all values of the time. In the case of the major planets it is always very small, the greatest value of $\sin^2\dfrac{I}{2}$ being for Mercury and Mars, 0.0118. In the perturbations of the planetoids by Jupiter it often fails, for I is sometimes of considerable magnitude while $r_2 - r_1$ may become very small. In the case of Mars and Eros $r_2 - r_1$ may actually vanish and this mode of development consequently fails. It is needless to say that it is not generally applicable in the cometary orbits.

In those cases in which the expansion of (84) does not fail, the expression for $R_{1,2}$ becomes

$$(87)\qquad \left\{\begin{aligned} R_{1,2} = &+[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{1}{2}}\\ &-r_1r_2[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{3}{2}}\\ &\qquad\qquad\times 2\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\frac{I}{2}\\ &+r_1^2r_2^2[r_1^2+r_2^2-2r_1r_2\cos(u_1-u_2+\tau_2-\tau_1)]^{-\frac{5}{2}}\\ &\qquad\qquad\times 6\sin^2(u_1-\tau_1)\sin^2(u_2-\tau_2)\sin^4\frac{I}{2}\\ &+\cdots\\ &-\frac{r_1}{r_2^2}\cos(u_1-u_2+\tau_2-\tau_1)\\ &+\frac{2r_1}{r_2^2}\sin(u_1-\tau_1)\sin(u_2-\tau_2)\sin^2\frac{I}{2}. \end{aligned}\right.$$

223. (*b*) **Development of the Coefficients in powers of e_1 and e_2.** The radii r_1 and r_2 vary from $a_1(1-e_1)$ and $a_2(1-e_2)$ to $a_1(1+e_1)$ and $a_2(1+e_2)$ respectively. Let

$$(88)\qquad \left\{\begin{aligned} r_1 &= a_1(1+\rho_1),\\ r_2 &= a_2(1+\rho_2). \end{aligned}\right.$$

The angles u_1 and u_2 are expressed in terms of the true anomalies, v_1 and v_2, and the elements by (82) The true anomalies are equal to the mean anomalies plus the equations of the center, which may be denoted by w_1 and w_2 Let l_1 and l_2 represent the mean longitudes counted from the x-axis [Fig (62)], then

$$(89)\qquad \begin{cases} u_1 - \tau_1 = l_1 - \Omega_1 - \tau_1 + w_1, \\ u_2 - \tau_2 = l_2 - \Omega_2 - \tau_2 + w_2 \end{cases}$$

It follows from (81) that $R_{1,2}$ can be written in the form

$$R_{1,2} = F[a_1(1 + \rho_1), a_2(1 + \rho_2)],$$

where F is a homogeneous function of a_1 and a_2 of degree -1. Therefore

$$(90)\qquad R_{1,2} = \frac{1}{1 + \rho_2} F\left[a_1 + a_1 \frac{\rho_1 - \rho_2}{1 + \rho_2}, a_2\right].$$

The right member of this equation can be developed by Taylor's formula, giving

$$(91)\qquad \begin{aligned} R_{1,2} = \frac{1}{1 + \rho_2}\Bigg\{ & F(a_1, a_2) + \frac{\rho_1 - \rho_2}{1 + \rho_2} \frac{a_1}{1} \frac{\partial F(a_1, a_2)}{\partial a_1} \\ & + \left(\frac{\rho_1 - \rho_2}{1 + \rho_2}\right)^2 \frac{a_1^2}{1 \cdot 2} \frac{\partial^2 F(a_1, a_2)}{\partial a_1^2} + \cdots \Bigg\}. \end{aligned}$$

The expressions $\left(\frac{\rho_1 - \rho_2}{1 + \rho_2}\right)^i$ can be developed as power series in ρ_1 and ρ_2 But in Art 100, equation (62), ρ is given as a power series in e whose coefficients are cosines of multiples of the mean anomaly On making these expansions and substitutions in (91), $R_{1,2}$ can be arranged as a power series in e_1 and e_2 These operations are to be actually performed upon the separate terms of the series (87), so the resulting series is arranged according to powers of e_1, e_2, and $\sin^2 \frac{I}{2}$ The angles w_1 and w_2 also depend upon e_1 and e_2 respectively, but their developments will not be introduced until after the next step

224. (*c*) **Developments in Fourier Series.** The first term within the bracket of (91) is obtained by replacing r_1 and r_2 by a_1 and a_2 respectively in (87) The higher terms involve the derivatives of the first with respect to a_1 On referring to the explicit series in (87), it is seen that the development of the expressions of the type

$$(a_1a_2)^{\frac{\nu-1}{2}}[a_1^2 + a_2^2 - 2a_1a_2 \cos (u_1 - u_2 + \tau_2 - \tau_1)]^{-\frac{\nu}{2}},$$

where ν is an odd integer, must be considered

Let $u_1 - u_2 + \tau_2 - \tau_1 = \psi$. It is known from the theory of Fourier series when a_1 and a_2 are unequal, as is assumed, that $[a_1^2 + a_2^2 - 2a_1a_2 \cos \psi]^{-\frac{\nu}{2}}$ can be developed into a series of cosines of multiples of ψ, which is convergent for all values of ψ. That is,

$$(92) \quad (a_1a_2)^{\frac{\nu-1}{2}}[a_1^2 + a_2^2 - 2a_1a_2 \cos \psi]^{-\frac{\nu}{2}} = \frac{1}{2}\sum_{i=-\infty}^{+\infty} B_\nu^{(i)} \cos i\psi,$$

where $B_\nu^{(i)} = B_\nu^{(-i)}$.

The coefficients $B_\nu^{(i)}$ are of course given by Fourier's integral

$$B_\nu^{(i)} = \frac{1}{\pi}\int_0^{2\pi} (a_1a_2)^{\frac{\nu-1}{2}}[a_1^2 + a_2^2 - 2a_1a_2 \cos \psi]^{-\frac{\nu}{2}} \cos i\psi\, d\psi,$$

but the difficulty of finding the integral makes it advisable in this particular problem to proceed otherwise.

Let $z = e^{\sqrt{-1}\psi}$, where e represents the Napierian base. Then

$$2 \cos \psi = z + z^{-1}, \qquad 2 \cos i\psi = z^i + z^{-i}.$$

Suppose $a_2 > a_1$ and let $\frac{a_1}{a_2} = \alpha$, then (92) becomes

$$(93) \quad \frac{\alpha^{\frac{\nu-1}{2}}}{a_2}(1 + \alpha^2 - 2\alpha \cos \psi)^{-\frac{\nu}{2}} = \frac{1}{2}\sum_{i=-\infty}^{+\infty} B_\nu^{(i)} \cos i\psi$$

Let

$$(1 + \alpha^2 - 2\alpha \cos \psi)^{-\frac{\nu}{2}} = (1 - \alpha z)^{-\frac{\nu}{2}}(1 - \alpha z^{-1})^{-\frac{\nu}{2}} = \frac{1}{2}\sum_{i=-\infty}^{+\infty} b_\nu^{(i)} z^i;$$

therefore

$$(94) \qquad B_\nu^{(i)} = \frac{\alpha^{\frac{\nu-1}{2}}}{a_2} b_\nu^{(i)}$$

Since the absolute values of αz and αz^{-1} are less than unity for all real values of ψ, the factors $(1 - \alpha z)^{-\frac{\nu}{2}}$ and $(1 - \alpha z^{-1})^{-\frac{\nu}{2}}$ can be expanded by the binomial theorem into convergent power series in αz and αz^{-1}. The coefficient of z^i in the product of these series is $\frac{1}{2}b_\nu^{(i)}$, after which $B_\nu^{(i)}$ is obtained from (94). The general term of the product of the expansions is easily found to be

$$
(95)\quad \begin{aligned}
\tfrac{1}{2}b_\nu^{(i)} = \frac{\frac{\nu}{2}\left(\frac{\nu}{2}+1\right)\cdots\left(\frac{\nu}{2}+i-1\right)}{i!}\alpha^i \Bigg[1 + \frac{\frac{\nu}{2}}{1}\,\frac{\frac{\nu}{2}+i}{i+1}\alpha^2 \\
+ \frac{\frac{\nu}{2}\left(\frac{\nu}{2}+1\right)}{1\cdot 2}\cdot\frac{\left(\frac{\nu}{2}+i\right)\left(\frac{\nu}{2}+i+1\right)}{(i+1)(i+2)}\alpha^4 + \cdots \Bigg].
\end{aligned}
$$

In this manner the coefficients of $\rho_1^{j_1}\rho_2^{j_2}\left(\sin^2\frac{I}{2}\right)^k$ are developed in Fourier series in $\cos i(u_1 - u_2 + \tau_2 - \tau_1)$. But these functions are multiplied by the factors $\sin(u_1 - \tau_1)\sin(u_2 - \tau_2)$ raised to different powers [equation (87)]. These powers of sines are to be reduced to sines and cosines of multiples of the arguments, and the products formed with $\cos i(u_1 - u_2 + \tau_2 - \tau_1)$, and the reduction again made to sines and cosines of multiples of arcs. The final trigonometrical terms will have the form $\cos(j_1u_1 + j_2u_2 + k_1\tau_1 + k_2\tau_2)$, where j_1, j_2, k_1, and k_2 are integers. As a consequence of (89) this expression can be developed into

$$
(96)\left\{\begin{aligned}
&\cos(j_1l_1 + j_2l_2 - j_1\Omega_1 - j_2\Omega_2 + k_1\tau_1 + k_2\tau_2 + j_1w_1 + j_2w_2) \\
&= \cos(j_1l_1 + j_2l_2 - j_1\Omega_1 - j_2\Omega_2 + k_1\tau_1 + k_2\tau_2) \\
&\qquad\times\{\cos(j_1w_1)\cos(j_2w_2) - \sin(j_1w_1)\sin(j_2w_2)\} \\
&\quad - \sin(j_1l_1 + j_2l_2 - j_1\Omega_1 - j_2\Omega_2 + k_1\tau_1 + k_2\tau_2) \\
&\qquad\times\{\sin(j_1w_1)\cos(j_2w_2) + \cos(j_1w_1)\sin(j_2w_2)\}.
\end{aligned}\right.
$$

Since

$$
\left\{\begin{aligned}
l_1 &= \Omega_1 + \omega_1 + n_1(t_0 - T_1) + n_1(t - t_0) = n_1t + \epsilon_1, \\
l_2 &= \Omega_2 + \omega_2 + n_2(t_0 - T_2) + n_2(t - t_0) = n_2t + \epsilon_2,
\end{aligned}\right.
$$

the first factors of the terms in the right member of this equation are independent of e_1 and e_2. Cos (j_1w_1), etc., are to be expanded into power series in w_1 and w_2 by the usual methods. Now $w_1 = v_1 - M_1$, $w_2 = v_2 - M_2$, and these quantities were developed into power series in e_1 and e_2 [Art. 100, eq. (64)] whose coefficients were Fourier series with multiples of the mean anomaly as arguments. On substituting these series for w_1 and w_2 in the expansions of the second factors of the terms of the right member of (96), and reducing the powers of sines and cosines of the mean anomaly to sines and cosines of multiples of the mean anomaly, and multiplying by the factors

$$\cos(j_1l_1 + j_2l_2 - j_1\Omega_1 - j_2\Omega_2 + k_1\tau_1 + k_2\tau_2)$$

and

$$\sin (j_1 l_1 + j_2 l_2 - j_1 \Omega_1 - j_2 \Omega_2 + k_1 \tau_1 + k_2 \tau_2),$$

and again reducing to sines and cosines of multiples of the arguments, the expression (96) is developed as a power series in e_1 and e_2 whose coefficients are series in sines and cosines of sums of multiples of l_1, l_2, Ω_1, Ω_2, τ_1, τ_2, M_1, M_2. But $M_1 = l_1 - \pi_1$, $M_2 = l_2 - \pi_2$, therefore the arguments will be l_1, l_2, Ω_1, Ω_2, τ_1, τ_2, π_1, π_2, where τ_1 and τ_2 are functions of Ω_1, Ω_2, i_1, and i_2 defined by (83).

When the several expansions and reductions which have been described have all been made, $R_{1,2}$ will be developed in a power series in e_1, e_2, and $\sin^2 \frac{I}{2}$, the coefficients of which are series of sines and cosines of multiples of l_1, l_2, Ω_1, Ω_2, τ_1, τ_2, π_1, π_2, the coefficient of each trigonometric term depending upon the ratio of the major semi-axes. If the signs of Ω_1, Ω_2, π_1, π_2, τ_1, τ_2, ϵ_1, ϵ_2, and t are changed the value of $R_{1,2}$, as defined in (84), obviously is unchanged, therefore the expansion in question contains only cosines of the argument. Hence

$$(97)\quad \begin{cases} R_{1,2} = \Sigma C \cos D, \\ \quad D = j_1(n_1 t + \epsilon_1) + j_2(n_2 t + \epsilon_2) - j_1' \Omega_1 - j_2' \Omega_2 \\ \qquad\qquad + k_1 \tau_1 + k_2 \tau_2 + k_1' \pi_1 + k_2' \pi_2, \\ \quad C = f\left(a_1,\ a_2,\ e_1,\ e_2,\ \sin^2 \frac{I}{2}\right), \end{cases}$$

in which $j_1, \cdots, k_2'$ take all integral values, positive, negative, and zero, the summation being extended over all of these terms.

It is clear from the foregoing that the series for $R_{1,2}$ is very complicated and that much labor is required to expand it in any particular case. Leverrier has carried out the literal development of all terms up to the seventh order inclusive in e_1, e_2, $\sin^2 \frac{I}{2}$, and the length of the work is such that fifty-three quarto pages of the first volume of the *Annales de l'Observatoire de Paris* are required in order to write out the result.

225. Periodic Variations. It follows from equations (72) and (97) that the rates of change of the elements of m_1 are given by

$$
(98)\begin{cases}
\dfrac{d\Omega_1}{dt} = \dfrac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\sum\left\{\dfrac{\partial C}{\partial i_1}\cos D \right. \\
\qquad\qquad \left. - \left[k_1\dfrac{\partial \tau_1}{\partial i_1} + k_2\dfrac{\partial \tau_2}{\partial i_1}\right] C\sin D\right\}, \\
\dfrac{di_1}{dt} = \dfrac{-m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\sum\left\{j_1' - k_1\dfrac{\partial \tau_1}{\partial \Omega_1} - k_2\dfrac{\partial \tau_2}{\partial \Omega_1}\right\} \\
\qquad\qquad \times C\sin D \\
\qquad + \dfrac{m_2\tan\dfrac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{k_1' + j_1 + k_1\dfrac{\partial \tau_1}{\partial \pi_1} + k_2\dfrac{\partial \tau_2}{\partial \pi_1}\right\} \\
\qquad\qquad \times C\sin D, \\
\dfrac{d\pi_1}{dt} = \dfrac{m_2\tan\dfrac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{\dfrac{\partial C}{\partial i_1}\cos D - \left[k_1\dfrac{\partial \tau_1}{\partial i_1} + k_2\dfrac{\partial \tau_2}{\partial i_1}\right]\right. \\
\qquad\qquad \left. \times C\sin D\right\} + \dfrac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\dfrac{\partial C}{\partial e_1}\cos D, \\
\dfrac{da_1}{dt} = \dfrac{-2m_2}{n_1 a_1}\sum j_1 C\sin D, \\
\dfrac{de_1}{dt} = m_2\sqrt{1-e_1^2}\,\dfrac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum j_1 C\sin D \\
\qquad + \dfrac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\left\{k_1' + k_1\dfrac{\partial \tau_1}{\partial \pi_1} + k_2\dfrac{\partial \tau_2}{\partial \pi_1}\right\} C\sin D, \\
\dfrac{d\epsilon_1}{dt} = \dfrac{m_2\tan\dfrac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{\dfrac{\partial C}{\partial i_1}\cos D - \left[k_1\dfrac{\partial \tau_1}{\partial i_1} + k_2\dfrac{\partial \tau_2}{\partial i_1}\right]\right. \\
\qquad \left. \times C\sin D\right\} + m_2\sqrt{1-e_1^2}\,\dfrac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\dfrac{\partial C}{\partial e_1}\cos D \\
\qquad\qquad - \dfrac{2m_2}{n_1 a_1}\sum\dfrac{\partial C}{\partial a_1}\cos D.
\end{cases}
$$

The perturbations of the elements of the orbit of m_1 of the first order with respect to the mass m_2 are the integrals of these equations regarding the elements as constants in the right members. Similar terms must be added for each disturbing planet.

There are terms in $R_{1,2}$ of three classes (*a*) those in which $j_1 n_1 + j_2 n_2$ is distinct from zero and not small; (*b*) those in which $j_1 n_1 + j_2 n_2$ is very small, but distinct from zero, and (*c*) those in which $j_1 n_1 + j_2 n_2$ equals zero Denote the fact that $R_{1,2}$ contains these three sorts of terms by writing

$$R_{1,2} = \Sigma C_0 \cos D_0 + \Sigma C_1 \cos D_1 + \Sigma C_2 \cos D_2,$$

where the three sums in the right member include these three classes of terms respectively Hence the perturbations of the elements of m_1 by m_2 of the first order and of the first class are

$$(99)\left\{\begin{aligned}
&(\Omega_1^{(0,1)}) - (\Omega_1^{(0,1)})_{t_0} = \frac{m_2}{n_1 a_1^2 \sqrt{1-e_1^2}\sin i_1} \\
&\qquad \times \sum\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2} + \left[k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right]\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}\right\}, \\
&(i_1^{(0,1)}) - (i_1^{(0,1)})_{t_0} = \frac{m_2}{n_1 a_1^2 \sqrt{1-e_1^2}\sin i_1} \\
&\qquad \times \sum\left\{j_1' - k_1\frac{\partial \tau_1}{\partial \Omega_1} - k_2\frac{\partial \tau_2}{\partial \Omega_1}\right\}\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2} \\
&\quad - \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{k_1' + j_1 + k_1\frac{\partial \tau_1}{\partial \pi_1} + k_2\frac{\partial \tau_2}{\partial \pi_1}\right\}\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}, \\
&(\pi_1^{(0,1)}) - (\pi_1^{(0,1)})_{t_0} = \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2}\right. \\
&\qquad \left. + \left[k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right]\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}\right\} \\
&\qquad + \frac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\frac{\partial C_0}{\partial e_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2}, \\
&(a_1^{(0,1)}) - (a_1^{(0,1)})_{t_0} = \frac{2m_2}{n_1 a_1}\sum j_1\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}, \\
&(e_1^{(0,1)}) - (e_1^{(0,1)})_{t_0} = -m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum j_1\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2} \\
&\qquad - \frac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\left\{k_1' + k_1\frac{\partial \tau_1}{\partial \pi_1} + k_2\frac{\partial \tau_2}{\partial \pi_1}\right\}\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}, \\
&(\epsilon_1^{(0,1)}) - (\epsilon_1^{(0,1)})_{t_0} = \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{\frac{\partial C_0}{\partial i_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2}\right. \\
&\qquad \left. + \left[k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right]\frac{C_0 \cos D_0}{j_1 n_1 + j_2 n_2}\right\} \\
&\qquad + m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\frac{\partial C_0}{\partial e_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2} \\
&\qquad\qquad - \frac{2m_2}{n_1 a_1}\sum\frac{\partial C_0}{\partial a_1}\frac{\sin D_0}{j_1 n_1 + j_2 n_2}.
\end{aligned}\right.$$

These terms are purely periodic with periods $\frac{2\pi}{j_1n_1 + j_2n_2}$, and constitute the *periodic variations* Every element is subject to them, depending upon an infinity of such terms whose periods are different The larger $j_1n_1 + j_2n_2$ is, the shorter is the period of the term and in general the smaller is its coefficient

The method of representing the motion of the planets by a series of periodic terms is somewhat analogous to the epicycloid theory of Ptolemy, for each term alone is equivalent to the adding of a small circular motion to that previously existing. This theory is more complex than that of Ptolemy in that it adds epicycloid upon epicycloid without limit, it is simpler than that of Ptolemy in that it flows from one simple principle, the law of gravitation.

226. Long Period Variations. The letters j_1 and j_2 represent all positive and negative integers and zero Therefore, unless n_1 and n_2 are incommensurable j_1 and j_2 exist such that $j_1n_1 + j_2n_2 = 0$, where j_1 and j_2 are not zero But then D is a constant and the integral is not formed this way However, whether n_1 and n_2 are incommensurable or not, such a pair of numbers can be found that $j_1n_1 + j_2n_2$ is very small The corresponding term will be large unless its C is very small It is shown in a complete discussion of the development of $R_{1,2}$ that the order of C in e_1, e_2, $\sin^2\frac{I}{2}$ is at the least equal to the numerical value of $j_1 + j_2$ (see Tisserand's *Méc. Cél.*, vol I, p. 308) Since n_1 and n_2 are both positive, one of the numbers j_1, j_2 must be positive and the other negative in order that the sum $j_1n_1 + j_2n_2$ shall be small. The more nearly equal j_1 and j_2 are numerically the smaller the numerical value of $j_1 + j_2$ is, and consequently, the larger C will be. When the mean motions of the two planets are such that they are nearly commensurable with the ratio of n_1 to n_2 expressible in small integers, then large terms in the perturbations will arise from the presence of these small divisors The period of such a term is $\frac{2\pi}{j_1n_1 + j_2n_2}$, which is very great, whence the appellation *long period*. These terms are given by equations of the same form as (99), but with the restriction that $j_1n_1 + j_2n_2$ shall be very small

Geometrically considered, the condition that the periods shall be nearly commensurable with the ratio expressible in small integers means that the points of conjunction occur at nearly the

same part of the orbits with only a few other conjunctions intervening. The extreme case is that in which there are no conjunctions intervening, i. e., when j_1 and j_2 differ in numerical value by unity

The mean motions of Jupiter and Saturn are nearly in the ratio of five to two Consequently $j_1 = 2$, $j_2 = -5$ gives a long period term, and the order of the coefficient C is the absolute value of $2 - 5$, or 3 The cause of the long period inequality of Jupiter and Saturn was discovered by Laplace in 1784 in computing the perturbations of the third order in e_1 and e_2 The length of the period in the case of these two planets is about 850 years

227. Secular Variations. The expression D is independent of the time for all of those terms in which $j_1 = j_2 = 0$ The partial derivatives of D with respect to the elements are also independent of the time, hence, on taking these terms of (98) and integrating, it is found that

$$
(100)\quad \left\{
\begin{aligned}
[\Omega_1^{(0,1)}] &= \frac{m_2}{n_1 a_1^2 \sqrt{1-e_1^2}\sin i_1}\sum\left\{\frac{\partial C_2}{\partial i_1}\cos D_2\right. \\
&\qquad \left. -\left[k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right]C_2 \sin D_2\right\}(t-t_0), \\
[i_1^{(0,1)}] &= \frac{-m_2}{n_1 a_1^2 \sqrt{1-e_1^2}\sin i_1}\sum\left\{j_1' - k_1\frac{\partial \tau_1}{\partial \Omega_1}\right. \\
&\qquad \left. - k_2\frac{\partial \tau_2}{\partial \Omega_1}\right\}C_2 \sin D_2 \cdot (t-t_0) \\
&\quad + \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{k_1' + k_1\frac{\partial \tau_1}{\partial \pi_1}\right. \\
&\qquad \left. + k_2\frac{\partial \tau_2}{\partial \pi_1}\right\}C_2 \sin D_2 \cdot (t-t_0), \\
[\pi_1^{(0,1)}] &= \frac{m_2 \tan\frac{i_1}{2}}{n_1 a_1^2\sqrt{1-e_1^2}}\sum\left\{\frac{\partial C_2}{\partial i_1}\cos D_2\right. \\
&\qquad \left. -\left[k_1\frac{\partial \tau_1}{\partial i_1} + k_2\frac{\partial \tau_2}{\partial i_1}\right]C_2 \sin D_2\right\}(t-t_0) \\
&\quad + \frac{m_2\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\sum\frac{\partial C_2}{\partial e_1}\cos D_2 \cdot (t-t_0),
\end{aligned}
\right.
$$

$$
(100)\left\{
\begin{aligned}
&[a_1^{(0,1)}] = 0,\\
&[e_1^{(0,1)}] = \frac{m_2\sqrt{1-e_1^2}}{n_1a_1^2e_1}\sum\left\{k_1' + k_1\frac{\partial\tau_1}{\partial\pi_1}\right.\\
&\qquad\qquad\qquad\qquad\left. + k_2\frac{\partial\tau_2}{\partial\pi_1}\right\}C_2\sin D_2\cdot(t-t_0),\\
&[\epsilon_1^{(0,1)}] = \frac{m_2\tan\frac{i_1}{2}}{n_1a_1^2\sqrt{1-e_1^2}}\sum\left\{\frac{\partial C_2}{\partial i_1}\cos D_2\right.\\
&\qquad\qquad\left. - \left[k_1\frac{\partial\tau_1}{\partial i_1} + k^2\frac{\partial\tau_2}{\partial i_1}\right]C_2\sin D_2\right\}(t-t_0)\\
&\qquad\qquad + m_2\sqrt{1-e_1^2}\,\frac{1-\sqrt{1-e_1^2}}{n_1a_1^2e_1}\\
&\qquad\qquad\qquad\qquad\times\sum\frac{\partial C_2}{\partial e_1}\cos D_2\cdot(t-t_0)\\
&\qquad\qquad - \frac{2m_2}{n_1a_1}\sum\frac{\partial C_2}{\partial a_1}\cos D_2\cdot(t-t_0).
\end{aligned}
\right.
$$

It follows that there are no secular terms of this type of the first order with respect to the masses in the perturbations of a. This constitutes the first theorem on the stability of the solar system. It was proved up to the second powers of the eccentricities by Laplace in 1773,* when he was but twenty-four years of age, in a memoir upon the mutual perturbations of Jupiter and Saturn, it was shown by Lagrange in 1776 that it is true for all powers of the eccentricities.† It was proved by Poisson in 1809 that there are no secular terms in a in the perturbations of the second order with respect to the masses, but that there are terms of the type $t \cos D$, where D contains the time.‡ Terms of this type are commonly called *Poisson terms*.

All of the elements except a have secular terms. It appears to have been supposed that the secular terms, which apparently cause the elements to change without limit, alone prevent the use of equations (72) for computing the perturbations for any time however great. Many methods of computing perturbations have been devised in order to avoid the appearance of secular terms; yet it is clear that, whether or not terms proportional to the time

* Memoir presented to the *Paris Academy of Sciences*.

† *Memoirs of the Berlin Academy*, 1776.

‡ *Journal de l'École Polytechnique*, vol. XV.

appear, the method is strictly valid for only those values of the time for which the series (20) of Art 207 are convergent

Secular terms may enter in another way, usually not considered. If $j_1 n_1 + j_2 n_2 = 0$ with $j_1 \neq 0$, $j_2 \neq 0$, D is independent of the time and the corresponding terms are secular In this case D is not independent of ϵ_1 and there will be secular terms in the perturbations of a. As has been remarked, this condition will always be fulfilled by an infinity of values of j_1 and j_2 if n_1 and n_2 are not incommensurable But it is impossible to determine from observations whether or not n_1 and n_2 are incommensurable, for there is always a limit to the accuracy with which observations can be made, and within this limit there exist infinitely many commensurable and incommensurable numbers There is as much reason, therefore, to say that secular terms in a of this type exist as that they do not. However, they are of no practical importance because the ratio of n_1 to n_2 cannot be expressed in small integers, and the coefficients of these terms, if they do exist, are so small that they are not sensible for such values of the time as are ordinarily used

228. Terms of the Second Order with Respect to the Masses. The terms of the second order are defined by equations (29), Art. 210. The right members of these equations are the products of the partial derivatives, with respect to the elements, of the right members which occur in the terms of the first order, and the perturbations of the first order of the corresponding elements Thus, the second order perturbations of the node are determined by the equations

$$(101)\quad \left\{\begin{aligned} \frac{d\Omega_1^{(0,2)}}{dt} &= \frac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\sum_{s_1}\frac{\partial^2 R_{1,2}}{\partial i_1 \partial s_1} s_1^{(0,1)}, \\ \frac{d\Omega_1^{(1,1)}}{dt} &= \frac{m_2}{n_1 a_1^2\sqrt{1-e_1^2}\sin i_1}\sum_{s_2}\frac{\partial^2 R_{1,2}}{\partial i_1 \partial s_2} s_2^{(1,0)}, \end{aligned}\right.$$

where s_1 and s_2 represent the elements of the orbits of m_1 and m_2 respectively. The partial derivative $\frac{\partial^2 R_{1,2}}{\partial i_1 \partial s_1}$ is a sum of periodic and constant terms; $s_1^{(0,1)}$ and $s_2^{(1,0)}$ are sums of periodic terms and terms containing the time to the first degree as a factor The products $\frac{\partial^2 R_{1,2}}{\partial i_1 \partial s_1} s_1^{(0,1)}$ and $\frac{\partial^2 R_{1,2}}{\partial i_1 \partial s_2} s_2^{(1,0)}$ therefore contain terms of

four types (a) $\frac{\sin}{\cos} D$, where D contains the time; (b) $t \frac{\sin}{\cos} D$; (c) $\frac{\sin}{\cos} D_2$, where D_2 is independent of the time, and (d) $t \frac{\sin}{\cos} D_2$. The integrals of these four types are respectively.

$$(a)\quad \frac{\frac{-\cos}{\sin} D}{j_1 n_1 + j_2 n_2}, \qquad (b)\quad t\frac{\frac{-\cos}{\sin} D}{j_1 n_1 + j_2 n_2} + \frac{\frac{\sin}{\cos} D}{(j_1 n_1 + j_2 n_2)^2};$$

$$(c)\quad t \frac{\sin}{\cos} D_2, \qquad (d)\quad \frac{t^2}{2} \frac{\sin}{\cos} D_2$$

Therefore, the perturbations of the second order with respect to the masses have purely periodic terms, Poisson terms, or terms in which the trigonometric terms are multiplied by the time; secular terms where the time occurs to the first degree, and secular terms where the time occurs to the second degree This is true for all of the elements except the major semi-axis, in the case of which the coefficients of the terms of the third and fourth types are zero, as Poisson first proved.

In the terms of the third order with respect to the masses there are secular terms in the perturbations of all the elements except a_1, which are proportional to the third power of the time, and so on.

229. Lagrange's Treatment of the Secular Variations. The presence of the secular terms in the expressions for the elements seems to indicate that, if it is assumed that the series represent the elements for all values of the time, then the elements change without limit with the time. But this conclusion is by no means necessarily true. For example, consider the function

$$(102)\qquad \sin(cmt) = cmt - \frac{c^3 m^3 t^3}{3!} + \cdots,$$

where c is a constant and m a very small factor which may take the place of a mass The series in the right member converges for all values of t. This function is never greater than unity for any value of the time, yet if its expansion in powers of m were given, and if the first few terms were considered without the law of the coefficients being known, it might seem that the series represents a function which increases indefinitely in numerical value with the time.

On following out the idea that the secular terms may be ex-

pansions of functions which are always finite, Lagrange has shown (see *Collected Works*, vols V and VI), under certain assumptions which have not been logically justified, that the secular terms are in reality the expansions of periodic terms of very long period These terms differ from the long period variations (Art 226) in that they come from the small uncompensated parts of the periodic variations, instead of directly from special conditions of conjunctions As a rule these terms are very small, and their periods are much longer than those of the sensible long period terms It will not be possible to give here more than a very general idea of the method of Lagrange

The first step in the method of Lagrange is a transformation of variables by the equations

$$(103) \qquad \begin{cases} h_j = e_j \sin \pi_j, \\ l_j = e_j \cos \pi_j, \end{cases}$$

and

$$(104) \qquad \begin{cases} p_j = \tan i_j \sin \Omega_j, \\ q_j = \tan i_j \cos \Omega_j, \end{cases}$$

where e_j, π_j, etc., are the elements of the orbit of m_j, and l_j is a new variable not to be confused with the mean longitude. These transformations are to be made simultaneously in the elements of the orbits of all of the planets. The elements a_j and ϵ_j remain without transformation. On omitting the subscripts, it is found from (103) and (104) that

$$(105) \qquad \begin{cases} \dfrac{dh}{dt} = +e \cos \pi \dfrac{d\pi}{dt} + \sin \pi \dfrac{de}{dt}, \\[2ex] \dfrac{dl}{dt} = -e \sin \pi \dfrac{d\pi}{dt} + \cos \pi \dfrac{de}{dt}, \\[2ex] \dfrac{\partial R}{\partial e} = \dfrac{\partial R}{\partial h}\dfrac{\partial h}{\partial e} + \dfrac{\partial R}{\partial l}\dfrac{\partial l}{\partial e} = \sin \pi \dfrac{\partial R}{\partial h} + \cos \pi \dfrac{\partial R}{\partial l}, \\[2ex] \dfrac{\partial R}{\partial \pi} = \dfrac{\partial R}{\partial h}\dfrac{\partial h}{\partial \pi} + \dfrac{\partial R}{\partial l}\dfrac{\partial l}{\partial \pi} = e \cos \pi \dfrac{\partial R}{\partial h} - e \sin \pi \dfrac{\partial R}{\partial l}, \\[2ex] \dfrac{dp}{dt} = +\tan i \cos \Omega \dfrac{d\Omega}{dt} + \sec^2 i \sin \Omega \dfrac{di}{dt}, \\[2ex] \dfrac{dq}{dt} = -\tan i \sin \Omega \dfrac{d\Omega}{dt} + \sec^2 i \cos \Omega \dfrac{di}{dt}, \end{cases}$$

$$
(105)\quad \begin{cases}
\dfrac{\partial R}{\partial \Omega} = \dfrac{\partial R}{\partial p}\dfrac{\partial p}{\partial \Omega} + \dfrac{\partial R}{\partial q}\dfrac{\partial q}{\partial \Omega} \\
\qquad = \tan \imath \cos \Omega \dfrac{\partial R}{\partial p} - \tan \imath \sin \Omega \dfrac{\partial R}{\partial q}, \\
\dfrac{\partial R}{\partial \imath} = \dfrac{\partial R}{\partial p}\dfrac{\partial p}{\partial \imath} + \dfrac{\partial R}{\partial q}\dfrac{\partial q}{\partial \imath} \\
\qquad = \sec^2 \imath \sin \Omega \dfrac{\partial R}{\partial p} + \sec^2 \imath \cos \Omega \dfrac{\partial R}{\partial q}.
\end{cases}
$$

Then it follows from (72) that

$$
(106)\quad \begin{cases}
\dfrac{dh}{dt} = \dfrac{m_2\sqrt{1-h^2-l^2}}{na^2}\dfrac{\partial R}{\partial l} \\
\qquad - \dfrac{m_2\sqrt{1-h^2-l^2}}{na^2}\dfrac{h}{1+\sqrt{1-h^2-l^2}}\dfrac{\partial R}{\partial \epsilon} \\
\qquad + \dfrac{m_2 l \tan \dfrac{i}{2}}{na^2\sqrt{1-h^2-l^2}}\dfrac{\partial R}{\partial i}, \\
\dfrac{dl}{dt} = \dfrac{-m_2\sqrt{1-h^2-l^2}}{na^2}\dfrac{\partial R}{\partial h} \\
\qquad - \dfrac{m_2\sqrt{1-h^2-l^2}}{na^2}\dfrac{l}{1+\sqrt{1-h^2-l^2}}\dfrac{\partial R}{\partial \epsilon} \\
\qquad - \dfrac{m_2 h \tan \dfrac{i}{2}}{na^2\sqrt{1-h^2-l^2}}\dfrac{\partial R}{\partial \imath}, \\
\dfrac{dp}{dt} = \dfrac{m_2}{na^2\sqrt{1-h^2-l^2}\cos^3 \imath}\dfrac{\partial R}{\partial q} \\
\qquad - \dfrac{m_2 p}{2na^2\sqrt{1-h^2-l^2}\cos \imath \cos^2 \dfrac{\imath}{2}}\left[\dfrac{\partial R}{\partial \pi} + \dfrac{\partial R}{\partial \epsilon}\right], \\
\dfrac{dq}{dt} = \dfrac{-m_2}{na^2\sqrt{1-h^2-l^2}\cos^3 i}\dfrac{\partial R}{\partial p} \\
\qquad - \dfrac{m_2 q}{2na^2\sqrt{1-h^2-l^2}\cos \imath \cos^2 \dfrac{\imath}{2}}\left[\dfrac{\partial R}{\partial \pi} + \dfrac{\partial R}{\partial \epsilon}\right].
\end{cases}
$$

On developing the right members of these equations and neglecting all terms of degree higher than the first* in h, l, p, and q, these

* The terms of order higher than the first are neglected throughout in a later step in the method

equations reduce to

$$(107)\qquad \left\{\begin{aligned} \frac{dh}{dt} &= +\frac{m_2}{na^2}\frac{\partial R}{\partial l}, \\ \frac{dl}{dt} &= -\frac{m_2}{na^2}\frac{\partial R}{\partial h}, \\ \frac{dp}{dt} &= +\frac{m_2}{na^2}\frac{\partial R}{\partial q}, \\ \frac{dq}{dt} &= -\frac{m_2}{na^2}\frac{\partial R}{\partial p}. \end{aligned}\right.$$

The terms which involve the derivative of R with respect to ϵ, i, and π do not appear in these equations because they involve h, l, p, or q as a factor. This fact follows from the properties of C given in Art. 226 and the form of equations (103) and (104).

Each perturbing planet contributes terms in the right members of equations (107) similar to the ones written which come from m_2. These differential equations are not strictly correct, since the first approximation has already been made in neglecting the higher powers of the variables.

The second step is in the method of treating the differential equations. The expansions of the $R_{i,j}$ contain certain terms which are independent of the time, which in the ordinary method give rise to the secular terms. Let $R^{(0)}_{i,j}$ represent these terms. Lagrange then treated the differential equations by neglecting the periodic terms in $R_{i,j}$, and writing

$$(108)\qquad \left\{\begin{aligned} \frac{dh_i}{dt} &= +\sum_{j=1}^{n} m_j \frac{\partial R^{(0)}_{i,j}}{\partial l_i}, \qquad (i = 1, \cdots, n;\ j \neq i), \\ \frac{dl_i}{dt} &= -\sum_{j=1}^{n} m_j \frac{\partial R^{(0)}_{i,j}}{\partial h_i}, \\ \frac{dp_i}{dt} &= +\sum_{j=1}^{n} m_j \frac{\partial R^{(0)}_{i,j}}{\partial q_i}, \\ \frac{dq_i}{dt} &= -\sum_{j=1}^{n} m_j \frac{\partial R^{(0)}_{i,j}}{\partial p_i}. \end{aligned}\right.$$

The values of h_i, l_i, p_i, and q_i determined from equations (108) are used instead of the secular terms obtained by the method of Art. 227. The process of breaking up a differential equation in this manner is not permissible except as a first approximation, and any conclusions based on it are open to suspicion.

In spite of the logical defects of the method and the fact that it cannot be generally applied, there is little doubt that in the present case it gives an accurate idea of the actual manner in which the elements vary

The right members of equations (108) are expanded in powers of h_i, l_i, p_i, and q_i, and all of the terms except those of the first degree are neglected, consequently the terms omitted in (107) would have disappeared here if they had been retained up to this point. The system becomes linear, and the detailed discussion of the $R_{i,j}$ shows that it is homogeneous, giving equations of the form

$$(109)\qquad \begin{cases} \dfrac{dh_1}{dt} - \sum\limits_{j=1}^{n} c_{1j} l_j = 0, \\ \dfrac{dl_1}{dt} + \sum\limits_{j=1}^{n} c_{1j} h_j = 0, \\ \dfrac{dh_2}{dt} - \sum\limits_{j=1}^{n} c_{2j} l_j = 0, \\ \dfrac{dl_2}{dt} + \sum\limits_{j=1}^{n} c_{2j} h_j = 0, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{dh_n}{dt} - \sum\limits_{j=1}^{n} c_{nj} l_j = 0, \\ \dfrac{dl_n}{dt} + \sum\limits_{j=1}^{n} c_{nj} h_j = 0, \end{cases}$$

and a similar system of equations in the p_j and the q_j.

The coefficients c_{ij} depend only on the major axes (the ϵ_j not appearing in the secular terms) which are considered as being constants, since the major axes have no secular terms in the perturbations of the first and second orders with respect to the masses. It is to be noted here that the assumption that the c_{ij} are constants is not strictly true because the major axes have periodic perturbations which may be of considerable magnitude

When these linear equations are solved by the method used in Art. 160, the values of the variables are found in the form

$$(110)\qquad \begin{aligned} h_i &= \sum_{j=1}^{n} H_{ij} e^{\lambda_j t}, & l_i &= \sum_{j=1}^{n} L_{ij} e^{\lambda_j t}, \\ p_i &= \sum_{j=1}^{n} P_{ij} e^{\mu_j t}, & q_i &= \sum_{j=1}^{n} Q_{ij} e^{\mu_j t}, \end{aligned}$$

where the H_{ij}, L_{ij}, P_{ij}, and Q_{ij} are constants depending upon the initial conditions. A detailed discussion shows that the λ_j and μ_j are all pure imaginaries with very small absolute values, therefore the h_j, l_j, p_j, and q_j oscillate around mean values with very long periods. Or, since the e_j and $\tan i_j$ are expressible as the sums of squares of the h_j, l_j, p_j, and q_j, it follows that they also perform small oscillations with long periods, for example, the eccentricity of the earth's orbit is now decreasing and will continue to decrease for about 24,000 years.

Equations (109) admit integrals first found by Laplace in 1784, which lead practically to the same theorem. They are

$$(111)\qquad \begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 (h_j^2 + l_j^2) = \text{Constant} = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 (p_j^2 + q_j^2) = C'; \end{cases}$$

or, because of (103) and (104),

$$(112)\qquad \begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 e_j^2 = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 \tan^2 i_j = C', \end{cases}$$

where n_j is the mean motion of m_j. The constants C and C' as determined by the initial conditions are very small, and since the left members of (112) are made up of positive terms alone, no e_j or i_j can ever become very great. There might be an exception if the corresponding m_j were very small compared to the others.

Equations (112) give the celebrated theorems of Laplace that the eccentricities and inclinations cannot vary except within very narrow limits. Although the demonstration lacks complete rigor, yet the results must be considered as remarkable and significant. Equations (112) do not give the periods and amplitudes of the oscillations as do equations (110).

230. Computation of Perturbations by Mechanical Quadratures. If the second term of the second factor of (84) in absolute value is greater than unity, the series (87) does not converge and cannot be used in computing perturbations. The expansions may fail because r_1 and r_2 are very nearly equal; or, sometimes when they are not nearly equal, because I is large. In the latter case

another mode of expansion sometimes can be employed,* but there are cases in which neither method leads to valid results. They both fail if the two orbits placed in the same plane would intersect, for in this case

$$r^2_{1,2} = r_1^2 + r_2^2 - 2r_1r_2 \cos (u_1 - u_2 + \tau_2 - \tau_1),$$

would vanish when the two bodies arrive at a point of intersection of their orbits at the same time Unless the periods are commensurable in a special way this would always happen. Of course, it is not necessary that $r_{1,2}$ should actually vanish in order that the expansion of (84) should fail to converge.

Perturbations can be computed by the method of mechanical quadratures without expanding the perturbative function explicitly in terms of the time Consequently, this method can be used in computing the disturbing effects of planets on comets and in other cases where the expansion of $R_{1,2}$ fails altogether or converges slowly Let s represent an element of the orbit of m_1, then equations (77) can be written in the form

$$\frac{ds}{dt} = f_s(t),$$

and the perturbations of the first order in the interval $t_n - t_0$ are

$$(113) \qquad s = s_0 + \int_{t_0}^{t_n} f_s(t)dt,$$

where s_0 is the value of s at $t = t_0$

The only difficulty in computing perturbations is in forming the integrals indicated in (113). When the perturbative function cannot be expanded explicitly in terms of t the primitive of the function $f_s(t)$ cannot be found. But in any case the values of $f_s(t)$ can be found for any values of t, and from the values of $f_s(t)$ for special values of t an approximation to the integral can be obtained. Geometrically considered, the integral (113) is the area comprised between the t-axis and the curve $f = f_s(t)$ and the ordinates t_0 and t_n An approximate value of the integral is

$$s \doteq s_0 + f_s(t_0)(t_1 - t_0) + f_s(t_1)(t_2 - t_1) + \cdots + f_s(t_{n-1})(t_n - t_{n-1}).$$

The intervals $t_1 - t_0$, $t_2 - t_1$, $\cdots$, $t_n - t_{n-1}$ can be taken so small that the approximation will be as close as may be desired.

Another method of obtaining an approximate value of the inte-

* Tisserand, *Mécanique Céleste*, vol. I, chap XXVIII.

gral is to replace the curve $f_s(t)$, whose explicit value in convenient form may not be obtainable, by a polynomial curve of the nth degree which agrees in value with $f_s(t)$ at $t = t_0, t_1, \cdots, t_n$. The equation of this polynomial is

$$
\begin{aligned}
f_s = &+ \frac{(t - t_1)(t - t_2)\cdots(t - t_n)}{(t_0 - t_1)(t_0 - t_2)\cdots(t_0 - t_n)} f_s(t_0) \\
&+ \frac{(t - t_0)(t - t_2)\cdots(t - t_n)}{(t_1 - t_0)(t_1 - t_2)\cdots(t_1 - t_n)} f_s(t_1) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&+ \frac{(t - t_0)(t - t_1)\cdots(t - t_{n-1})}{(t_n - t_0)(t_n - t_1)\cdots(t_n - t_{n-1})} f_s(t_n)
\end{aligned}
$$

Since there is no trouble in forming the integral of a polynomial there is no trouble in computing the perturbation of s for the interval $t_n - t_0$. If the value of the function $f_s(t)$ is not changing very rapidly or irregularly, its representation by a polynomial is very exact provided the intervals $t_1 - t_0, \cdots, t_n - t_{n-1}$ are not too great.

However, the area between the polynomial, the t-axis, and the limiting ordinates is not the best approximation to the value of the integral that can be obtained from the values of $f_s(t)$ at $t_0, \cdots, t_n$. The values of the function give information respecting the nature of the curvature of the curve between the ordinates (this being true, of course, only because the function $f_s(t)$ is a regular function of t), and corrections of the area due to these curvatures can easily be made. Ordinarily they would involve the derivatives of $f_s(t)$ at $t_0, \cdots, t_n$, which would require a vast amount of labor to compute; but the derivatives can be expressed with sufficient approximation in terms of the successive differences of the function, and the differences are obtained directly from the tabular values by simple subtraction. The derivation of the most convenient explicit formulas is a lengthy matter and must be omitted.*

Suppose the computation of the integrals from the values of $f_s(t)$ at $t = t_0, \cdots, t_n$ has not given results which are sufficiently exact. More exact ones can be obtained by dividing the interval $t_n - t_0$ into a greater number of sub-intervals. A little experience usually makes it unnecessary to subdivide the intervals first chosen.

* See Tisserand's *Mécanique Céleste*, vol. IV, chaps X and XI; and Charlier's *Mechanik des Himmels*, vol II., chap. 1

There is a second reason why the results obtained by mechanical quadratures may not be sufficiently exact. It has so far been assumed that $f_s(t)$ is a function of t alone, or, in other words, that the elements of the orbits on which it depends are constants. This is the assumption in computing perturbations of the first order. If it is not exact enough, new values of $f_s(t_1), \cdots, f_s(t_n)$ can be computed, on using in them the respective values of the elements s which were found by the first integration. From the new values of $f_s(t_1), \cdots, f_s(t_n)$ a more approximate value of the integral can be obtained. Unless the interval $t_n - t_0$ is too great this process converges and the integral can be found with any desired degree of approximation, because this method is simply Picard's method of successive approximations whose validity has been established.* In practice it is always advisable to choose the interval $t_n - t_0$ so short that no repetition of the computation with improved values of the function at the ends of the sub-intervals will be required. At each new stage of the integration the values of the elements at the end of the preceding step are employed. It follows that the method, as just explained, enables one to compute not only the perturbations of the first order, but perturbations of all orders except for the limitations that the intervals cannot be taken indefinitely small and the computation cannot be made with indefinitely many places.

The process of computing perturbations by the method of mechanical quadratures, as compared with that of using the expanded form of the perturbative function, has its advantages and its disadvantages. It is an advantage that in employing mechanical quadratures it is not necessary to express the perturbing forces explicitly in terms of the elements and the time. This is sometimes of great importance, for, in cases where the eccentricities and inclinations are large, as in some of the asteroid orbits, these expressions, which are series, are very slowly convergent; and in the case of orbits whose eccentricities exceed 0.6627, or of orbits which have any radius of one equal to any radius of the other the series are divergent and cannot be used. The method of mechanical quadratures is equally applicable to all kinds of orbits, the only restriction being that the intervals shall be taken sufficiently short. It is the method actually employed, in one of its many forms, in computing the perturbations of the orbits of comets.

* Picard's *Traité d'Analyse*, vol. II, chap. XI, section 2.

The disadvantages are that, in order to find by mechanical quadratures the values of the elements at any particular time, it is necessary to compute them at all of the intermediate epochs. Being purely numerical, it throws no light whatever on the general character of perturbations, and leads to no general theorems regarding the stability of a system These are questions of great interest, and some of the most brilliant discoveries in Celestial Mechanics have been made respecting them

231. General Reflections. Astronomy is the oldest science and in a certain sense the parent of all the others The relatively simple and regularly recurring celestial phenomena first taught men, in the days of the ancient Greeks, that Nature is systematic and orderly The importance of this lesson can be inferred from the fact that it is the foundation on which all science is based For a long time progress was painfully slow Centuries of observations and attempts at theories for explaining them were necessary before it was finally possible for Kepler to derive the laws which are a first approximation to the description of the way in which the planets move. The wonder is that, in spite of the distractions of the constant struggles incident to an unstable social order, there should have been so many men who found their greatest pleasure in patiently making the laborious observations which were necessary to establish the laws of the celestial motions

The work of Kepler closed the preliminary epoch of two thousand years, or more, and the brilliant discoveries of Newton opened another. The invention of the Calculus by Newton and Leibnitz furnished for the first time mathematical machinery which was at all suitable for grappling with such difficult problems as the disturbing effects of the sun on the motion of the moon, or the mutual perturbations of the planets. It was fortunate that the telescope was invented about the same time, for, without its use, it would not have been possible to have made the accurate observations which furnished the numerical data for the mathematical theories and by which they were tested The history of Celestial Mechanics during the eighteenth century is one of a continuous series of triumphs The analytical foundations laid by Clairaut, d'Alembert, and Euler formed the basis for the splendid achievements of Lagrange and Laplace. Their successors in the nineteenth century pushed forward, by the same methods on the whole, the theories of the motions of the moon and planets to higher orders of approximation and compared them with more

and better observations. In this connection the names of Leverrier, Delaunay, Hansen, and Newcomb will be especially remembered Near the close of the nineteenth century a third epoch was entered. It is distinguished by new points of view and new methods which, in power and mathematical rigor, enormously surpass all those used before It was inaugurated by Hill in his *Researches on the Lunar Theory*, but owes most to the brilliant contributions of Poincaré to the Problem of Three Bodies

At the present time Celestial Mechanics is entitled to be regarded as the most perfect science and one of the most splendid achievements of the human mind. No other science is based on so many observations extending over so long a time. In no other science is it possible to test so critically its conclusions, and in no other are theory and experience in so perfect accord There are thousands of small deviations from conic section motion in the orbits of the planets, satellites, and comets where theory and the observations exactly agree, while the only unexplained irregularities (probably due to unknown forces) are a very few small ones in the motion of the moon and the motion of the perihelion of the orbit of Mercury. Over and over again theory has outrun practise and indicated the existence of peculiarities of motion which had not yet been derived from observations. Its perfection during the time covered by experience inspires confidence in following it back into the past to a time before observations began, and into the future to a time when perhaps they shall have ceased As the telescope has brought within the range of the eye of man the wonders of an enormous space, so Celestial Mechanics has brought within reach of his reason the no lesser wonders of a correspondingly enormous time. It is not to be marveled at that he finds profound satisfaction in a domain where he is largely freed from the restrictions of both space and time.

XXVII. PROBLEMS.

1 Suppose (*a*) that $R_{1,2}$ is large and nearly constant, (*b*) that $R_{1,2}$ is large and changing rapidly, (*c*) that $R_{1,2}$ is small and nearly constant If the perturbations are computed by mechanical quadratures how should the $t_n - t_0$ be chosen relatively in the three cases, and how should the numbers of subdivisions of $t_n - t_0$ compare?

2. The perturbative function involves the reciprocal of the distance from the disturbing to the disturbed planets This is called the *principal part* and gives the most difficulty in the development. How many separate reciprocal

distances must be developed in order to compute, in a system of one sun and n planets, (a) the perturbations of the first order of one planet, (b) the perturbations of the first order of two planets, (c) the perturbations of the second order of one planet, and (d) the perturbations of the third order of one planet?

3 What simplifications would there be in the development of the perturbative function if the mutual inclinations of the orbits were zero, and if the orbits were circles?

4 What sorts of terms will in general appear in perturbations of the third order with respect to the masses?

HISTORICAL SKETCH AND BIBLIOGRAPHY

The theory of perturbations, as applied to the Lunar Theory, was developed from the geometrical standpoint by Newton The memoirs of Clairaut and D'Alembert in 1747 contained important advances, making the solutions depend upon the integration of the differential equations in series Clairaut soon had occasion to apply his processes of integration to the perturbations of Halley's comet by the planets Jupiter and Saturn This comet had been observed in 1531, 1607, and 1682 If its period were constant it would pass the perihelion again about the middle of 1759 Clairaut computed the perturbations due to the attractions of Jupiter and Saturn, and predicted that the perihelion passage would be April 13, 1759 He remarked that the time was uncertain to the extent of a month because of the uncertainties in the masses of Jupiter and Saturn and the possibility of perturbations from unknown planets beyond these two The comet passed the perihelion March 13, giving a striking proof of the value of Clairaut's methods

The theory of the perturbations of the planets was begun by Euler, whose memoirs on the mutual perturbations of Jupiter and Saturn gained the prizes of the French Academy in 1748 and 1752 In these memoirs was given the first analytical development of the method of the variation of parameters The equations were not entirely general as he had not considered the elements as being all simultaneously variables The first steps in the development of the perturbative function were also given by Euler

Lagrange, whose contributions to Celestial Mechanics were of the most brilliant character, wrote his first memoir in 1766 on the perturbations of Jupiter and Saturn In this work he developed still further the method of the variation of parameters, leaving his final equations, however, still incorrect by regarding the major axes and the epochs of the perihelion passages as constants in deriving the equations for the variations The equations for the inclination, node, and longitude of the perihelion from the node were perfectly correct In the expressions for the mean longitudes of the planets there were terms proportional to the first and second powers of the time These were entirely due to the imperfections of the method, their true form being that of the long period terms, as was shown by Laplace in 1784 by considering terms of the third order in the eccentricities The method of the variation of parameters was completely developed for the first time in 1782 by Lagrange in a prize memoir on the perturbations of comets moving in

elliptical orbits By far the most extensive use of the method of variation of parameters is due to Delaunay, whose Lunar Theory is essentially a long succession of the applications of the process, each step of it removing a term from the perturbative function

In 1773 Laplace presented his first memoir to the French Academy of Sciences In it he proved his celebrated theorem that, up to the second powers of the eccentricities, the major axes, and consequently the mean motions of the planets, have no secular terms This theorem was extended by Lagrange in 1774 and 1776 to all powers of the eccentricities and of the sine of the angle of the mutual inclination, for perturbations of the first order with respect to the masses Poisson proved in 1809 that the major axes have no purely secular terms in the perturbations of the second order with respect to the masses Haretu proved in his Dissertation at Sorbonne in 1878 that there *are* secular variations in the expressions for the major axes in the terms of the third order with respect to the masses In vol XIX of *Annales de l'Observatoire de Paris*, Eginitis considered terms of still higher order with respect to the masses

Lagrange began the study of the secular terms in 1774, introducing the variables h, l, p, and q The investigations were carried on by Lagrange and Laplace, each supplementing and extending the work of the other, until 1784 when their work became complete by Laplace's discovery of his celebrated equations

$$\begin{cases} \sum_{j=1}^{n} m_j n_j a_j^2 e_j^2 = C, \\ \sum_{j=1}^{n} m_j n_j a_j^2 \tan^2 i_j = C'. \end{cases}$$

These equations were derived by using only the linear terms in the differential equations. Leverrier, Hill, and others have extended the work by methods of successive approximations to terms of higher degree Newcomb (*Smithsonian Contributions to Science*, vol XXI, 1876) has established the more far-reaching results that it is possible, in the case of the planetary perturbations, to represent the elements by purely periodic functions of the time which formally satisfy the differential equations of motion If these series were convergent the stability of the solar system would be assured; but Poincaré has shown that they are in general divergent (*Les Méthodes Nouvelles de la Mécanique Céleste*, chap. IX) Lindstedt and Gyldèn have also succeeded in integrating the equations of the motion of n bodies in periodic series, which, however, are in general divergent

Gauss, Airy, Adams, Leverrier, Hansen, and many others have made important contributions to the planetary theory in some of its many aspects. Adams and Leverrier are noteworthy for having predicted the existence and apparent position of Neptune from the unexplained irregularities in the motion of Uranus More recently Poincaré turned his attention to Celestial Mechanics, publishing a prize memoir in the *Acta Mathematica*, vol XIII This memoir was enlarged and published in book form with the title *Les Méthodes Nouvelles de la Mécanique Céleste* Poincaré applied to the problem all the resources of modern mathematics with unrivaled genius, he brought into the investigation such a wealth of ideas, and he devised methods of such immense power

that the subject in its theoretical aspects has been entirely revolutionized in his hands It cannot be doubted that much of the work of the next fifty years will be in amplifying and applying the processes which he explained

The following works should be consulted

Laplace's *Mécanique Céleste*, containing practically all that was known of Celestial Mechanics at the time it was written (1799–1805)

On the variation of parameters—*Annales de l'Observatoire de Paris*, vol I, Tisserand's *Mécanique Céleste*, vol I, Brown's *Lunar Theory*, Dziobek's *Planeten-Bewegungen*

On the development of the perturbative function—*Annales de l'Observatoire de Paris*, vol I, Tisserand's *Mécanique Céleste*, vol I, Hansen's *Entwickelung des Products einer Potenz des Radius-Vectors mit dem Sinus oder Cosinus eines Vielfachen der wahren Anomalie*, etc, *Abh d K Sächs Ges zu Leipzig*, vol II, Newcomb's memoir on the General Integrals of Planetary Motion, Poincaré, *Les Méthodes Nouvelles*, vol I, chap VI

On the stability of the solar system—Tisserand's *Mécanique Céleste*, vol I, chaps XI, XXV, XXVI, and vol IV, chap XXVI, Gylden, *Traité Analytique des Orbites absolues*, vol I, Newcomb, *Smithsonian Cont*, vol XXI, Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, vol II, chap X

On the subject of Celestial Mechanics as a whole there is no better work available than that of Tisserand, which should be in the possession of every one giving special attention to this subject. Another noteworthy work is Charlier's *Mechanik des Himmels*, which, besides maintaining a high order of general excellence, is unequaled by other treatises in its discussion of periodic solutions of the Problem of Three Bodies.

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